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Introduction

This paper is concerned with the study of equations of the form

$$\text{(1)} \qquad Lu = \frac{1}{i}\frac{du}{dt} - Au = 0$$

for functions $u(t)$ with values in some Banach space, as well as inhomogeneous equations

$$\text{(2)} \qquad Lu = f$$

and slightly perturbed equations, with the main emphasis on the behavior of solutions as $t \to +\infty$.

In case A is the generator of a semi-group such equations have been studied in great detail, see the book by E. Hille and R. Phillips [1]. We mention also the book by Lions [1] which describes much of the recent work on equations (1), (2). We shall, however, treat equations for which the initial value problem (prescribing u at some value of t) is not necessarily well posed. In particular, we consider equations arising from partial differential equations in a cylinder (with t axis along the generator) which may be elliptic, and for which, therefore, the initial value problem is indeed not well posed. The operator A then represents a partial differential operator in the variables in the base of the cylinder. Many of the questions treated here grew out of topics considered by P. D. Lax in a series of papers [1-3].

In connection with the applications to differential equations in a cylinder it is convenient to suppose that $u(t)$ lies in a Banach space X and that Lu lies in Y, $X \subset Y$ and $| \ |_X \geq | \ |_Y$. The operator A is assumed to be a closed operator with domain D_A in X and range in Y. By a solution of (2) we shall mean a function $u(t)$ such that (i) $u(t) \in D_A$ for every t under consideration; (ii) $u(t)$, as an element in X, is strongly continuous in t and, as an element in Y, strongly differentiable, with $Du = \frac{1}{i}\frac{du}{dt}$ strongly continuous in Y; (iii) $Lu = f$ holds. It will be clear that it will often suffice to assume that Du is absolutely integrable in every compact interval. Furthermore, in case X and Y are Hilbert spaces and L_2 estimates are considered, the solution may (often) only be a generalized solution with $|u|_X$ and $|Du|_Y$ square integrable. However we shall not bother to point out such generalizations.

Chapters 1 to 4 are concerned with the general theory; applications to partial differential equations in a cylinder being made in Chapter 5. In the general theory we treat four topics, all but Chapter 4, which is concerned with the regularity of solutions of (2), involving the behavior of solutions at infinity. Other, natural, questions - such as construction of fundamental solutions - are not treated. In this connection see H. Tanabe [1] in which (2) is considered with $A = A(t)$ a function of t. For every t, $A(t)$ is the generator of a semigroup, and, under suitable conditions, Tanabe constructs a fundamental solution for the equation. Throughout the general theory we make constant use of

the following methods: Fourier transform, and Parseval's theorem in case X, Y are Hilbert spaces, some elementary complex variable theory, in particular, Phragmén-Lindelöf theorems and contour deformations; we also use the Paley-Wiener theorem. In each chapter we present a variety of related results only a few of which are applied in Chapter 5. These are given to illustrate the techniques employed. Indeed the techniques are often, to us, of greater interest than the results, and it is hoped that they will be suggestive to others, and that some of the ideas indicated here will be developed further - especially for operators varying with t. All the arguments and results employed here in the general theory are fairly elementary, with the possible exception of the Mihlin multiplier theorem of §3.

The main assumptions throughout the paper are concerned with the region of regularity in the complex λ plane of the resolvent of A

$$R(\lambda) = R(\lambda;A) = (\lambda I - A)^{-1}$$

considered as a map of Y into X, as well as conditions on the behavior at infinity in this region of the norm $|R(\lambda)|_X$ (or $|R(\lambda)|_Y$) as a bounded mapping from Y into X (or Y). (When considering solutions on the semi-infinite line $t > 0$, and blowing up at most exponentially, the behavior of $R(\lambda)$ in only a half plane $\operatorname{Im} \lambda >$ constant enters, while in studying solutions on a finite interval we use information about $R(\lambda)$ in regions in upper and lower half planes.) In the applications we study operators

of higher order which we reduce to first order operators L by introducing the derivatives with respect to t of a function as new unknowns. In doing so the range of the corresponding first order operator L acting on the function u, and its derivatives will in general be confined to a subspace S of Y and one is usually interested in the behavior of $(\lambda I - A)^{-1}$ restricted just to this subspace. For this reason, in the abstract theory, we introduce also the restriction $R_S(\lambda)$ of $R(\lambda)$ to a closed subspace S of Y, and postulate that $R_S(\lambda)$ be holomorphic with respect to λ on its region of existence. A more precise motivation for the introduction of the operator $R_S(\lambda)$ is given in §2.

Chapter 1 is concerned with stability at $t \to +\infty$ of solutions of (2); we consider a slightly perturbed equation expressed in the form

$$|Lu|_X \leq \phi(t)|u|_X + b(t) \tag{3}$$

where $b(t)$, $\phi(t)$ are scalar valued functions and $\phi(t)$ tends to zero with some rapidity. Assuming, say, that b dies down exponentially and that $|u|_X$ belongs to L_2 we give conditions assuring that $|u|_X$ then also dies down exponentially. Thus we prove, in the notation of Lax [3] an abstract Phragmén-Lindelöf principle. Here we assume (Theorem 1.4) that X, Y are Hilbert spaces and that $R(\lambda)$ (or $R_S(\lambda)$) is regular and bounded in a strip $0 \leq \operatorname{Im} \lambda < a$ in the complex plane except for a finite number of poles on the real axis; the function ϕ is then required to decay like t^{-k} where k is the maximal order of the poles.

In Chapter 2 we assume that $R(\lambda)$ is meromorphic in the upper half plane (or in a suitable large strip therein) and derive in §4 and §5 asymptotic formulas for solutions of (1) for $t > 0$ as an infinite series of exponential solutions. We shall call a solution of (1) of the form $\chi(t) = e^{i\lambda_o t} p(t)$, where $p(t)$ is a polynomial in t with coefficients in X, an exponential solution. It is easily seen that a necessary and sufficient condition for $\chi(t)$ to be a solution with

$$p(t) = \phi_m + it\phi_{m-1} + \dots + \frac{(it)^{m-1}}{(m-1)!}\,\phi_1 , \qquad \text{with} \quad \phi_1 \neq 0 ,$$

is that λ_o is an eigenvalue of A, and $(A - \lambda_o)\phi_1 = 0$, $(A - \lambda_o)\phi_j = \phi_{j-1}$, $j = 2,\dots,m$; m is called the index of the exponential solution. It follows that $\chi_k(t) = e^{i\lambda_o t} p_k(t)$, for $k = 1,\dots,m-1$, are also solutions, where

$$p_k(t) = \phi_{m-k} + it\phi_{m-k-1} + \dots + \frac{(it)^{m-1-k}}{(m-1-k)!}\,\phi_1 .$$

The functions χ_k are called the associates of χ. It is readily seen that the m-dimensional space spanned by χ and its associates coincides with the space spanned by $\chi(t)$ and its derivatives $\chi^{(j)}(t)$; this is also the space spanned by the translates $\chi(t+s)$ of $\chi(t)$.

Under suitable conditions the asymptotic formulas are proved to hold also for certain complex values of t (Theorem 2.3, where it is shown to be analytic in t). These enable us (Theorem 2.4) to give lower bounds for $|u(t)|_Y$ showing that solutions cannot decay with arbitrary speed.

In §6 we present a result called an abstract Weinstein principle which asserts, under suitable conditions that any solution of (1) on the entire line $-\infty < t < \infty$ blowing up at most exponentially at $\pm\infty$ is a finite sum of exponential solutions.

In §7 we discuss completeness on $t > 0$ of exponential solutions belonging say, to L_2 among all L_2 solutions. Here R is assumed to be meromorphic in the upper half plane, and a notion of lower order of $R(\lambda)$ is employed. We also consider completeness of all exponential solutions among solutions on a finite interval, assuming $R(\lambda)$ to be meromorphic in the whole plane.

Chapters 1 and 2 should be read together.

In Chapter 3, entitled "unique continuation and lower bounds at infinity" we attempt tp show under various conditions that a solution of

$$(3)' \qquad |Lu|_X \leq \phi(t)|u|_X$$

cannot decay too rapidly at infinity unless it is identically zero for large t. In §8 we consider the finite "backward" Cauchy problem: assuming $u(T) = 0$ for some T we show that $u(t) = 0$ for $t < T$. In §9 we present some results ensuring that a solution u of (3)' which is $O(e^{at})$ as $t \to +\infty$ for every a is identically zero for large t. In the remaining §§10, 11 of Chapter 3 we give lower bounds for solutions (via convexity arguments) under various hypotheses.

Chapter 3 is essentially self-contained and may be read independently of the remainder of the paper; the results are not applied in Chapter 5 since the theorems they yield for partial

differential equations seem rather special. We hope however that the techniques will encourage further research in this direction.

Chapter 4, which is also self-contained, is concerned with the differentiability or analyticity of solutions of (2) assuming f to be differentiable or analytic. We prove necessary conditions, and present sufficient conditions which are not far removed from the necessary ones. The necessary conditions are obtained by employing the "closed graph theorem" in a suitable space, while the proofs of sufficiency use Fourier transform and elementary complex variable. The results should be extended to equations in which A is allowed to vary with t.

In Chapter 5 we apply the abstract theory to a class of partial differential operators in a cylinder which we term "weighted elliptic". These include elliptic operators as well as a large class of parabolic operators. The basic estimate which enables us to apply the preceding theory is given in Theorem 5.4. In proving completeness of exponential solutions we also use Theorem 5.4' which is based on a recent result of Agmon [2]. In §§13-16 we present the basic properties of weighted elliptic operators, which are derived from results for elliptic boundary value problems. Section 15 contains some remarks that are of interest for general elliptic boundary value problems in a bounded domain. In §§17, 18 we investigate asymptotic series and completeness of exponential solutions for equations with coefficients independent of t, as $t \to \infty$, and in §19 we consider the stability question of exponential decay in slightly perturbed equations. The

results are used to indicate how one might show that the space of solutions of a homogeneous elliptic boundary value problem in an unbounded domain — the solutions satisfying some growth condition at infinity — is finite dimensional. An example is given at the end of §17 showing that in general this need not be the case.

We wish to express our thanks to Peter Lax for many stimulating and illuminating comments and to Lars Hörmander for a number of useful suggestions.

Chapter I

Stability at Infinity and Exponential Decay

1. A preliminary result

In this chapter we are interested in the behavior near infinity of solutions of (3), the $|\ |_X$ norms of the solutions being assumed to belong to L_p for some $p \geq 1$ (in case X and Y are Hilbert spaces we may set $p = 2$), or to L^0_∞ (bounded measurable and tending to zero as $t \to \infty$). In particular, we seek conditions on A to ensure exponential decay (in L_p) of solutions of (3) with $b = 0$, i.e. an abstract Phragmén-Lindelöf principle. We shall present several results in this direction.

We begin with an abstract version of Theorem 2.2 of Lax [3] - having essentially the same proof. In this theorem we permit A to depend on t, $A = A(t)$ and then require that $u(t)$ belong to $D_{A(t)}$ for every t.

Theorem 1.1: Suppose that for every function $v(t)$ vanishing at $t = 0$, with $|v(t)|_X$ and $|Lv(t)|_Y$ belonging to L_p on $t > 0$, the inequality

$$(1.1) \qquad \big||v(t)|_X\big|_{L_p} \leq C\big||Lv|_Y\big|_{L_p}$$

holds, for some $p \geq 1$ and some fixed constant C. Here $|b|_{L_p}$ represents the L_p norm (in t) of the scalar valued function $b(t)$. Let $u(t)$ be a solution of

$$(1.2) \qquad |Lu(t)|_Y \leq c|u(t)|_X + b(t) , \qquad t > 0$$

with $|u(t)|_X \in L_p$. *If* $c < C^{-1}$ *and if* $|e^{at}b(t)|_{L_p} < \infty$ *for some positive number* λ *then for every positive number* $\sigma \leq a$ *satisfying* $(\sigma + c)C < 1$ *the* L_p *norm of* $e^{\sigma t}|u(t)|_X$ *is finite and, in fact,*

$$\int_1^\infty e^{p\sigma t}|u(t)|_X^p \, dt \leq k_1 \int_0^1 |u(t)|_Y^p \, dt + k_2 |e^{\sigma t} b(t)|_{L_p}^p ,$$

where k_1, k_2 *are constants depending only on* C, c *and* σ.

Proof: For any number $T \geq 1$ let $\tau(t) \leq t$ be a real, continuously differentiable, monotonic function of t which equals t for $t \leq T$, is constant for $t \geq T+1$, and satisfies $\frac{d\tau}{dt} \leq 1$. Let $\zeta(t)$ be a continuously differentiable monotonic function vanishing at the origin, and equal to one for $t \geq 1$, with $\frac{d\zeta}{dt} \leq 2$. If we apply (1.1) to the function $v = \zeta(t)e^{\sigma\tau}u(t)$ we obtain the inequality

$$\begin{aligned}
||\zeta e^{\sigma\tau}u|_X|_{L_p} &\leq C||L(\zeta e^{\sigma\tau}u)|_Y|_{L_p} \\
&\leq C|\zeta e^{\sigma\tau}|Lu|_Y|_{L_p} + C|\tfrac{d\zeta}{dt}\, e^{\sigma\tau}|u|_Y|_{L_p} + C\sigma|\zeta e^{\sigma\tau}|u|_Y|_{L_p} \\
&\leq Cc|\zeta e^{\sigma\tau}|u|_X|_{L_p} + C|\zeta e^{\sigma\tau} b|_{L_p} \\
&\qquad + 2Ce^{\sigma}\left[\int_0^1 |u(t)|_Y^p \, dt\right]^{1/p} + C\sigma|\zeta e^{\sigma\tau}|u|_X|_{L_p}
\end{aligned}$$

by (1.2). Hence

$$1 - C(\sigma + c)\ |\zeta e^{\sigma\tau}|u|_X|_{L_p} \leq C|e^{\sigma t}b|_{L_p} + 2Ce^{\sigma}\left[\int_0^1 |u|_Y^p \, dt\right]^{1/p} .$$

Since this is true for any T we find on letting $T \longrightarrow \infty$, so that $\tau \longrightarrow t$,

$$(1 - C(\sigma + c))|\zeta e^{\sigma t}|u|_X|_{L_p} \leq \text{same right-hand side} ,$$

from which the desired result follows.

In subsequent theorems of Phragmén-Lindelöf type we shall make hypotheses concerning the resolvent $R(\lambda) = (\lambda I - A)^{-1}$ of A, regarded as a mapping from Y into X. We note here that (1.1) yields some estimates for the norm of $(\lambda - A)^{-1}$ (for convenience we shall omit writing the identity operator I). For instance, if λ is real and u is any vector in D_A then, for any differentiable function $\zeta(t)$ with compact support, we have, by (1.1),

$$|\zeta|u|_X|_{L_p} = |e^{i\lambda t}\zeta|u|_X|_{L_p} \leq C||L(e^{i\lambda t}\zeta u)|_Y|_{L_p}$$

$$\leq C||\zeta(\lambda - A)u|_Y|_{L_p} + C|\frac{d\zeta}{dt}\ |u|_Y|_{L_p} .$$

Thus

$$|u|_X|\zeta(t)|_{L_p} \leq C|(\lambda - A)u|_Y|\zeta|_{L_p} + C|u|_X|\frac{d\zeta}{dt}|_{L_p} .$$

Choosing for ζ a function which vanishes at the origin, is equal to one on a long interval, and then goes down to zero again, we find easily that

$$|u|_X \leq C|(\lambda - A)u|_Y .$$

From this it follows that if some real number belongs to the resolvent set of A then every real number does, and the norm of the resolvent $R(\lambda)$, as a mapping from Y into X, is bounded by C for real λ. (In case X and Y are Hilbert spaces, and p = 2, one

sees, with the aid of Plancherel's theorem that such a bound on the resolvent implies, in turn, (1.1).)

2. Equations of higher order

In the following sections we wish our results to be applicable also to differential equations of arbitrary order. As we shall see a certain difficulty arises. Consider an equation of order ℓ of the form

$$(2.1) \qquad \tilde{L}u = D^{\ell}u + \sum_{j=1}^{\ell} A_j D^{\ell-j}u = f$$

where the A_j are operators. We have in mind the situation where the A_j are differential operators of orders jd, $j = 1,\ldots,\ell$, for some integer d, acting on functions of some other variables. In this context it usually makes sense to require that the different derivatives $D^j u(t)$ belong to different spaces $\tilde{B}_j$, $j = 0,\ldots,m$, with $\tilde{B}_0 \subset \tilde{B}_1 \subset \ldots \subset \tilde{B}_\ell$.

We shall thus consider Banach spaces $\tilde{B}_j$, $j = 0,\ldots,\ell$, $\tilde{B}_0 \subset \tilde{B}_1 \subset \ldots \subset \tilde{B}_\ell$ with norms $\tilde{|}u|_j \geq \tilde{|}u|_{j+1}$, $j = 0,\ldots,\ell-1$, and assume that each A_j is a closed operator with domain in $\tilde{B}_{\ell-j}$ and range in $\tilde{B}_\ell$, $j = 1,\ldots,\ell$. We also assume that there is a constant K such that

$$\tilde{|}A_j u|_\ell \leq K \tilde{|}u|_{\ell-j}, \qquad \text{for} \quad j = 1,\ldots,\ell-1,$$

so that these operators are continuous. As a function of t, u(t) is to be strongly continuous in $\tilde{B}_0$ and, as an element in $\tilde{B}_1$, to be strongly differentiable with Du strongly continuous in $\tilde{B}_1$ and,

generally, $D^j u$ strongly continuous in $\tilde{B}_j$ for $j \leq \ell$; also $D^{\ell-j}u$ is supposed to be in the domain of A_j.

Equation (2.1) may be written as a first order system in the usual way by introducing new dependent variables

$$u_j = D^j u \,, \qquad j = 0,\ldots,\ell-1$$

the system being

$$(2.1)' \qquad \begin{aligned} &Du_j - u_{j+1} = 0 \,, \qquad j = 0,\ldots,\ell-2 \\ &Du_{\ell-1} + \sum_{j=1}^{\ell} A_j u_{\ell-j} = f \,. \end{aligned}$$

Setting now $U = (u_o,\ldots,u_{\ell-1})$ the general inhomogeneous system $LU = DU - AU = F$, $F = (f_o,\ldots,f_{\ell-1})$, takes the form

$$(2.2) \qquad \begin{aligned} &Du_j - u_{j+1} = f_j \,, \qquad j = 0,\ldots,\ell-2 \\ &Du_{\ell-1} + \sum_{j=0}^{\ell-1} A_{\ell-j} u_j = f_{\ell-1} \,, \end{aligned}$$

the operator A being defined in this system. Here $u_j(t)$ is strongly continuous in $\tilde{B}_j$, in the domain $A_{\ell-j}$, and with $Du_j(t)$ strongly continuous in $\tilde{B}_{j+1}$. We observe that if U is a solution of (2.2) then its components satisfy

$$(2.3) \qquad \begin{aligned} &\tilde{L}u_o = \sum_{k=1}^{\ell-1} \left[D^{\ell-k} + \sum_{j=k}^{\ell-1} A_{\ell-j} D^{j-k}\right] f_{k-1} + f_{\ell-1} \,, \\ &u_j = D^j u_o - \sum_{k=1}^{j} D^{j-k} f_{k-1} \,, \qquad j > 0 \,. \end{aligned}$$

Set $X = \tilde{B}_o \times \tilde{B}_1 \times \ldots \times \tilde{B}_{\ell-1}$, $Y = \tilde{B}_1 \times \tilde{B}_2 \times \ldots \times \tilde{B}_\ell$, with norms $|U|_X = \sum_0^{\ell-1} \tilde{|} u_j|_j$, $|U|_Y = \sum_0^{\ell-1} \tilde{|} u_j|_{j+1}$.

Let us now attempt with the aid of Theorem 1.1 to derive an analogue of the theorem for the operator $\tilde{L}$. An obvious analogue of (1.1) is the assumption that for every v with $\tilde{|} D^j v(t)|_j$ vanishing at the origin, for $j < \ell$, and belonging to L_p for $t > 0$, with $\tilde{|}\tilde{L}v|_\ell$ in L_p the inequality

$$(2.4) \qquad \Big|\sum_{j=0}^{\ell-1} \tilde{|} D^j v|_j\Big|_{L_p} \leq C\big|\tilde{|}\tilde{L}v|_\ell\big|_{L_p}$$

holds for some $p \geq 1$.

<u>Theorem 1.1'</u>: <u>Let</u> u(t) <u>be a solution of the differential inequality</u>

$$(2.5) \qquad \tilde{|}\tilde{L}u(t)|_\ell \leq c \sum_{j=0}^{\ell-1} \tilde{|} D^j u(t)|_j + b(t) ,$$

<u>with</u> $cC < 1$, <u>and assume that the right-hand side of</u> (2.5) <u>belongs to</u> L_p <u>for</u> $t > 0$, <u>while</u> $e^{at}b(t)$ <u>belongs to</u> L_p <u>for some</u> $a > 0$. <u>Then there exist positive constants</u> k_1, k_2 <u>and</u> $\sigma \leq a$ <u>depending only on</u> C, c, m <u>and</u> K <u>such that for</u> $T \geq 1$

$$e^{\sigma T} \int_T^\infty \Big(\sum_{j=0}^{\ell-1} \tilde{|} D^j u|_j\Big)^p dt \leq k_1 \int_0^1 \Big(\sum_{j=0}^{\ell-1} \tilde{|} D^j u(t)|_j\Big)^p dt + k_2 |e^{\sigma t} b(t)|_{L_p}^p .$$

Theorem 1.1' may be proved in the same manner as Theorem 1.1. However if we attempt to derive it directly by applying Theorem 1.1 to L with A mapping a domain in X into Y, we see that

condition (1.1) is not satisfied, i.e. (2.4) does not imply (1.1). By (2.3) we see that inequality (2.4) implies (1.1) only for those $v(t)$ in the domain of L such that the first $m-1$ components of Lv vanish. This suggests modifying Theorem 1.1 in a suitable way so that we consider only functions $u(t)$ for which $Lu(t)$ lies in a certain linear subspace of Y. But then the proof of the theorem presented above will not work, for at one point we apply inequality (1.1) to $\zeta(t)$ times the function $u(t)$, and if Lu belongs to the required subspace the function $L(\zeta u)$ need not. Thus we are led to make the following

Hypothesis: Let S be a closed subspace of Y. Assume that there is an operator $\zeta\cdot$ defined for functions $u(t)$ on $t \geq 0$, with $Lu(t)$ in S for every t, such that:

(i) $v(t) = (\zeta\cdot u)(t)$ equals $u(t)$ for $t \geq 1$,

(ii) $v(t)$ vanishes for $t \leq 0$,

(iii) $Lv(t)$ also lies in S ,

(iv) $|Lv|_Y \leq \kappa(|Lu|_Y + |u|_Y)$, for $0 \leq t \leq 1$,

where κ is a fixed constant independent of u.

In extending Theorem 1.1 we remark that it is often useful in practice to consider solutions of more general differential inequalities

$$(2.6) \quad |Lu(t)|_Y \leq c \sum_1^N |P_j u(t)|_X + c \sum_1^N |Q_j u(t)|_Y + b(t) , \quad t > 0 ,$$

where each P_j (Q_j) is an operator with domain in X containing the domain of A, and with range in X (Y). We wish then to prove exponential decay not only for $|u(t)|_X$ but also for $|P_j u(t)|_X$

and $|Q_j u(t)|_Y$. Naturally we shall have to assume a stronger form of inequality (1.1). We shall always assume that one of the P_j, say P_1, is the identity operator, and shall use the notation

$$(2.6)' \qquad \|u(t)\| = \sum |P_j u(t)|_X + \sum |Q_j u(t)|_Y \,.$$

We now formulate our modification of Theorem 1.1.

Theorem 1.1": Let S be a closed subspace of Y and let u(t) be a solution of (2.6) such that Lu(t) lies in S for every t. Assume also that for every function v(t), with v(0) = 0 and Lv(t) in S, such that $|Lv(t)|_Y$ and $\|v(t)\|$ belong to L_p on t > 0, the following inequality holds: for some $p \geq 1$ and some fixed constant C.

$$(2.7) \qquad \big|\, \|v(t)\| \,\big|_{L_p} \leq C \big|\, |Lv(t)|_Y \big|_{L_p} \,.$$

Assume that $|e^{at} b(t)|_{L_p} < \infty$ for some positive number a. Under our hypothesis of a $\zeta\cdot$ operator, if $cC < 1$, there exist positive constants σ, k_1, k_2 depending only on C, c and κ such that for $T \geq 1$

$$e^{\sigma T} \int_T^{\infty} \|u(t)\|^p \, dt \leq k_1 \int_0^1 \|u(t)\|^p \, dt + k_2 |e^{\sigma t} b(t)|_{L_p}^p \,.$$

Theorem 1.1' follows from Theorem 1.1" if we set $P_1 = I$ and all other $P_j = Q_k = 0$, and take for S the vectors whose first m-1 components vanish, and define $\zeta\cdot U = (\rho u_o, D(\rho u_o), \ldots, D^{m-1}(\rho u_o))$; here $\rho(t)$ is a nonnegative monotonic C^{∞} function vanishing for $t \leq 0$ and equal to one for $t \geq 1$.

The proof of Theorem 1.1" will be slightly different from that of Theorem 1.1. It is based on the following well known lemma whose proof we omit - though the lemma will be used repeatedly.

Lemma 1.1: If $\alpha(t)$, $\beta(t)$ are continuous nonnegative decreasing functions for $t > 0$ satisfying

$$\alpha(t) \leq \beta(t) + c'(\alpha(t-1) - \alpha(t)) , \qquad t \geq 1 ,$$

for some positive constant c', then for $t \geq 1$,

$$e^{\sigma t}\alpha(t) \leq c_1(\alpha(0) - \alpha(1)) + c_1 \int_1^\infty e^{\sigma t}\beta(t)dt$$

where c_1, σ are constants depending only on c' - provided the right-hand side is finite.

Proof of Theorem 1.1": In virtue of the lemma it suffices to show that

$$\int_T^\infty \|u(t)\|^p \, dt \leq k' \int_{T-1}^\infty |b(t)|^p \, dt + k' \int_T^{T+1} \|u(t)\|^p \, dt$$

for some constant $k' = k'(C,c,\kappa)$. This inequality follows (by translation) from the inequality for $T = 1$; thus we consider only $T = 1$. Setting $v = \zeta \cdot u$ and applying (2.7) we find that

$$\int_1^\infty \|u(t)\|^p\, dt \le \int_0^\infty \|v\|^p\, dt \le C^p \int_0^\infty |Lv(t)|_Y^p\, dt$$

$$\le C^p \int_1^\infty |Lu|_Y^p\, dt + C^p \kappa^p \int_0^1 (|Lu|_Y + |u|_Y)^p\, dt \quad \text{by (iv)}$$

$$\le C^p \int_1^\infty (c\|u\| + b)^p\, dt + C^p \kappa^p \int_0^1 ((c+1)\|u\| + b)^p\, dt$$

by (2.6). Since $cC < 1$ we conclude that

$$\int_1^\infty \|u\|^p\, dt \le k' \int_0^\infty b^p\, dt + k' \int_0^1 \|u\|^p\, dt .$$

This is the desired result for $T = 1$.

Note: Throughout the remainder of the paper whenever we consider functions $u(t)$ with Lu in S we shall assume that our $\zeta\cdot$ hypothesis holds. We shall also assume this to be true for operators of the form $L + aI$ with any constant a.

The set of values λ such that $(\lambda - A)^{-1}$ is a bounded map defined on all of S into X will be called the S-resolvent set ρ_S of A; its complement is called the S-spectrum of A. On ρ_S, $(\lambda-A)^{-1}$ will be denotes by $R_S(\lambda) = R_S(\lambda;A)$. If S were all of Y we could assert, in the usual way that $R_S(\lambda)$ is an analytic operator valued function of λ in ρ_S. However, for $S \neq Y$ this may not be the case. Nevertheless, for the situation we have in mind - of a first order system derived from an equation of order m, where S consists of vectors whose first m-1 components vanish - $R_S(\lambda)$ is analytic. Therefore we shall postulate that $R_S(\lambda)$ is analytic in ρ_S.

We shall denote by $|R_S(\lambda)|_X$ $(|R_S(\lambda)|_Y)$ the norm of $R_S(\lambda)$ as a mapping from S into X (Y), dropping the subscripts whenever $X = Y = S$.

Returning again to the operator $\tilde{L}$ we may obtain as before, the following consequence of (2.4): for $u \in \tilde{B}_0$ and in the domain of each operator A_j, the following holds:

$$\sum_{j=0}^{\ell-1} \tilde{|}\lambda^j u|_j \leq \text{constant } \tilde{|}(\lambda^\ell + \sum_1^\ell A_j \lambda^{\ell-j})u|_\ell , \qquad \lambda \text{ real} .$$

If now A is the operator occurring in the system (2.2) and if $(\lambda - A)U = F$, λ real, it follows that

$$|U|_X \leq \text{constant } (K+1)\left[\tilde{|}f_{\ell-1}|_\ell + \sum_{k=1}^{\ell-1} \sum_{j=k}^{\ell} |\lambda|^{j-k}|f_{k-1}|_j\right] .$$

Thus

$$|U|_X \leq \text{constant } (K+1)|F|_Y \quad \text{if } F \text{ is in our space } S ,$$

while in general

$$|U|_X \leq \text{constant } (K+1)(1 + |\lambda|^{m-1})|F|_Y \quad \text{for } \lambda \text{ real.}$$

We see therefore that if we do not restrict ourselves to S the resolvent may grow like a polynomial on the real axis. In some of our subsequent results this additional growth will be harmless, and we shall then state our results without considering S, i.e. by permitting S to be all of Y.

When considering (2.6) we shall denote $P_j R_S(\lambda)$ by $P_{jS}(\lambda)$, or simply by $P_j(\lambda)$ if $S = Y$; if it is a bounded operator of S into X its norm will be denoted by $|P_{jS}(\lambda)|_X$. Similarly

$Q_j R_S(\lambda) = Q_{jS}(\lambda)$, or simply $Q_j(\lambda)$ if $S = Y$; its norm as mapping from S into Y will be denoted by $|Q_{jS}(\lambda)|_Y$. We set

$$(2.8) \qquad K(\lambda) = \sum |P_{jS}(\lambda)|_X + \sum |Q_{jS}(\lambda)|_Y .$$

In practice when A is a differential operator acting in some function space the operators P_j, Q_j will also be differential operators - of lower order in general. For simplicity we shall state our results without the operators P_j, Q_j - indicating by remarks afterwards how the results can be extended to include them.

The behavior of the resolvent $R_S(\lambda)$ will play an important role in our analysis of solutions u of Lu = 0. It may occur that for some linear operator P whose domain contains the domain of A, and with range in X (or Y) that $PR(\lambda)$ can be extended as a regular analytic (bounded) operator valued function in a region in the complex plane which is larger than ρ_S. This will be reflected in the behavior of Pu. (Again we have in mind the case where X is a space of functions defined on some domain in a Euclidean space, and Pu is, say, a restriction of the function u to some subdomain.)

3. Stability at infinity

In this section we assume that X and Y are Hilbert spaces. We use the notation of §2 (see in particular (2.8)). Our first extension of Theorems 1.1 and 1.1' is

Theorem 1.2: Assume that $R_S(\lambda)$ exists for all λ in a strip $-\varepsilon < \text{Im}\,\lambda < a$, for ε, $a > 0$, except possibly for a finite number of real points λ_i, $i = 1,\dots,m$ which are poles of $R_S(\lambda)$, and assume

that the norm $|R_S(\lambda)|_X = O(1)$ for $|\lambda| \to \infty$ in the strip. (For instance if $S = Y$ this will follow if $R_S(\lambda)$ exists on the whole real axis except for the poles at λ_i, and has bounded norm $|R_S(\lambda)|_X$ as $|\lambda| \to \infty$ on the axis.) Let k denote the maximal order of the poles. Let $u(t)$ be a solution with $|u(t)|_X$ in L_2 for $t > 0$ of

$$(3.1) \qquad |Lu|_Y \leq \frac{c}{(1+t)^k} |u|_X + b(t)$$

where c is a constant and $b(t)$ is a function such that $e^{a't}b(t)$ belongs to L_2 for some positive $a' < a$.

Conclusion: There exists a positive number c' depending only on the operator A such that if $c \leq c'$ then $|e^{a't}u|_X$ belongs to L_2; in fact

$$(3.2) \quad \int_0^\infty (e^{a't}|u|_X)^2\, dt \leq C \int_0^1 |u|_X^2\, dt + C \int_0^\infty |e^{a't}b(t)|^2\, dt$$

where C is a constant depending only on A, a' (and the constant κ of (iv) in §2). If we merely assume that $(1+t^k)b(t) \in L_2$ then

$$(3.2)_1 \quad \int_0^\infty |u(t)|_X^2\, dt \leq C \int_0^1 |u(t)|_X^2\, dt + C \int_0^\infty |(1+t)^k b(t)|^2\, dt\ .$$

It is clear that it suffices to assume (3.1) only for t sufficiently large in order to prove the exponential decay. The theorem is sharp in the sense that if c is not small then u need not decay exponentially. This may be shown quite generally but we mention here only the following simple counterexample. Suppose that $X = Y$ is one dimensional and consider the differential equation

$$\frac{du}{dt} = -\frac{c}{1+t}\,u\,, \qquad\qquad t > 0\,.$$

Here we suppose $A = 0$ so that $R(\lambda)$ has a pole at the origin of first order, c is a positive constant. Thus a solution $u = K(1+t)^{-c}$ certainly satisfies (3.1) with $b = 0$. If $c > \frac{1}{2}$ the solution belongs to L_2 but does not die down exponentially.

In proving the theorem we shall establish (3.2) with a' replaced by a small number a''. Then, by repeating the argument for the function $v = e^{a''t}u$, which satisfies

$$|Dv-(A-ia'')v|_Y \leq \frac{c}{(1+t)^k}\,|v|_X + e^{a''t}b(t)\,,$$

we deduce that $e^{a'''t}|v|_X$ belongs to L_2 for some $a''' > 0$. Repeating the argument again (applied to $e^{a'''t}|u|_X$ belongs to L_2, and the corresponding inequality (3.2) is easily established. The theorem is related to classical results by Dunkel [1]. Its proof is based on

<u>Lemma 1.2</u>: (a) <u>Assume that</u> $R_S(\lambda)$ <u>satisfies the conditions in Theorem 1.2</u>. <u>For any nonnegative integer</u> n <u>there exists a constant</u> c_n <u>depending only on</u> n <u>and</u> A <u>such that any function</u> $v(t)$, <u>vanishing at the origin, for which</u> $|v|_X$, $(1+t^{n+k})|Lv|_Y$ <u>are square integrable on</u> $t > 0$, (<u>as usual we require</u> $Lv(t)$ <u>to lie in</u> S) <u>satisfies</u>

$$(3.3) \qquad |(1+t^n)|v|_X|_{L_2} \leq c_n|(1+t^{n+k})|Lv|_Y|_{L_2}\,.$$

<u>In particular the left-hand side is finite.</u>

(b) <u>Assume in addition that the norm</u> $|R_S(\lambda)|_Y$ <u>is</u> $O(\frac{1}{\lambda})$ <u>as</u> $|\lambda| \to \infty$ <u>in the strip</u>. <u>Then also</u>

$$(3.4) \qquad |(1+t^n)|Dv|_Y|_{L_2} \leq c_n'|(1+t^{n+k})|Lv|_Y|_{L_2} .$$

_If_ $R_S(\lambda)$ _has at most one pole in the strip at the origin then in fact_

$$(3.4)' \quad |(1+t^{n+1})|Dv|_Y|_{L_2} \leq c_n'|(1+t^{n+k'})|Lv|_Y|_{L_2} , \qquad k' = \max(1,k).$$

Here c_n' _depends only on_ A _and_ n.

Before proving the lemma we shall use part (a) to prove (3.2) for small a', with c' any positive number $< c_o^{-1}$. We shall use σ, C_1, C_2 to denote constants depending only on c_o, k and κ. Recalling the $\zeta\cdot$ operator of §2 we define for fixed $T \geq 1$

$$v(t) = (\zeta\cdot u(t+T-1))(t) .$$

By Lemma 1.2 (a) and the properties (i)-(iv) of $\zeta\cdot$ we have

$$\begin{aligned}
\int_T^\infty |u(t)|_X^2\,dt &\leq ||v|_X|_{L_2}^2 \leq c_o^2 \int_0^\infty (1+t^k)^2|Lv|_Y^2\,dt \\
&\leq c_o^2 \int_1^\infty (1+t^k)|Lu(t+T-1)|_Y^2\,dt \\
&\quad + c_o^2\kappa^2 \int_0^1 (1+t^k)^2(|Lu(t+T-1)|_Y + |u(t+T-1)|_Y)^2\,dt \\
&\leq c_o^2 \int_T^\infty (1+t^k)^2|Lu(t)|_Y^2\,dt \\
&\quad + c_1 \int_{T-1}^T (|Lu(t)|_Y^2 + |u(t)|_Y^2)dt .
\end{aligned}$$

Using (3.1) we find now that this is

$$\leq c_o^2 \int_T^\infty (c|u(t)|_X + (1+t^k)b(t))^2\, dt + C_2 \int_{T-1}^T (|u(t)|_X^2 + b^2(t))dt\ .$$

Since $c \leq c' < c_o^{-1}$ it follows easily that

$$(3.2)' \quad \int_T^\infty |u(t)|_X^2\, dt \leq C_3 \int_{T-1}^T |u(t)|_X^2\, dt + C_3 \int_{T-1}^\infty (1+t^k)^2 b^2(t)dt\ .$$

Thus $(3.2)_1$ is established.

We may now apply Lemma 1,1 and infer that for some positive constant σ (we may assume $\sigma < a$), and every $T \geq 1$,

$$e^{\sigma T} \int_T^\infty |u(t)|_X^2\, dt \leq C_4 \int_0^1 |u(t)|_X^2\, dt$$

$$+ C_4 \int_1^\infty e^{\sigma t}\, dt \int_{t-1}^\infty (1+\tau^k)^2 b^2(\tau)d\tau\ .$$

Thus for some (smaller) constant σ the function $w(t) = e^{\sigma t}u(t)$ is a solution of

$$|Dv - (A-i\sigma)v|_Y \leq \frac{c}{(1+t)^k} |v|_X + e^{\sigma t}b(t)\ ,$$

with square integrable $|\ |_X$ norm. Since the resolvent of $A-i\sigma$ is now regular and bounded on the real axis we find, by repeating the preceding argument (with now $k = 0$), the following analogue of (3.2)'

$$\int_1^\infty |w(t)|_X^2\,dt \le c_3 \int_0^1 |w(t)|_X^2\,dt + c_3 \int_0^\infty e^{2\sigma t} b^2(t)\,dt$$

from which (3.2) follows for $a' = \sigma$.

By a similar argument, using Lemma 1.2 (b) we may prove

<u>Theorem 1.2'</u>: <u>Assume that</u> $R_S(\lambda)$ <u>satisfies the conditions of Theorem</u> 1.2, <u>and that</u> $|R_S(\lambda)|_Y = O(\frac{1}{\lambda})$ <u>for</u> $|\lambda| \to \infty$ <u>in the strip</u>. <u>Let</u> $b(t)$ <u>be a function as in Theorem</u> 1.2. <u>Then</u> (a) <u>there exists a positive number</u> c'' <u>depending only on</u> A (<u>and</u> κ) <u>such that if</u> $u(t)$ <u>is a solution with</u> $|u(t)|_X$, $|Du|_Y$, $|Au|_Y$ <u>in</u> L_2 <u>for</u> $t > 0$ <u>of</u>

$$(3.1)' \qquad |Lu(t)|_Y \le \frac{c}{(1+t)^k} U(t) + b(t)$$

<u>where</u>

$$U = |u(t)|_X + |Du(t)|_Y + |Au(t)|_Y$$

<u>with</u> $c \le c''$ <u>then</u> $u(t)$ <u>satisfies a stronger form of</u> (3.2) <u>and</u> $(3.2)_1$

$$(3.2)'' \qquad \begin{aligned} \int_0^\infty |e^{a't}U|^2\,dt &\le c \int_0^1 |U|^2\,dt + c \int_0^\infty |e^{a't}b(t)|^2\,dt \\ \int_0^\infty |U|^2\,dt &\le c \int_0^1 |U|^2\,dt + c \int_0^\infty |(1+t)^k b(t)|^2\,dt\,. \end{aligned}$$

(b) <u>If furthermore</u>, $R_S(\lambda)$ <u>has only a pole at the origin of of order</u> $k \ge 1$ <u>then the same conclusion holds</u>, <u>with</u> U <u>replaced by</u> V <u>for a solution</u> $u(t)$, <u>with</u> $|u|_X$, $|Lu|_Y \in L_2$ <u>for</u> $t > 0$, <u>of</u>

$$(3.1)'' \qquad |Lu(t)|_Y \le \frac{c}{(1+t)^k} V + b(t)$$

<u>if</u> $c < c''$; <u>here</u>

$$V(t) = |u(t)|_X + (1+t)|Du(t)|_Y + (1+t)|Au(t)|_Y .$$

<u>Proof of Lemma 1.2</u>: (a) $C_1, C_2, \ldots$ will denote constants depending only on A. Setting $v(t) = 0$ fot $t < 0$ let $\hat{v}(\lambda)$ be the Fourier transform of $v(t)$:

$$\hat{v}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\lambda t} v(t)dt$$

and let $\hat{f}(\lambda)$ be the Fourier transform of $f(t) = Lv(t)$. Since Plancherel's Theorem holds for L_2 functions with values in a Hilbert space, $\hat{v}$, $\hat{f}$, and, in fact, $(\frac{d}{d\lambda})^{n+k}\hat{f}$ exist as L_2 functions with values in X and Y respectively. Since S is closed in Y it follows that $\hat{f}(\lambda)$ lies in S. We have, furthermore, for almost all real λ, $(\lambda - A)\hat{v}(\lambda) = \hat{f}(\lambda)$, so that

$$\hat{v}(\lambda) = R_S(\lambda)\hat{f}(\lambda) . \tag{3.5}$$

Denoting now $R_S(\lambda)$ simply by $R(\lambda)$ we intend to decompose $R(\lambda)$ as a finite sum $R(\lambda) = \sum_0^{m+1} R_j(\lambda)$, with $R_j(\lambda)$ equal to $R(\lambda)$ in a neighborhood of the pole λ_j, and vanishing near the other poles, and each R_j vanishing outside a finite interval, for $j = 1,\ldots,m$. Imagine the poles ordered $\lambda_1 < \lambda_2 < \ldots < \lambda_m$ and introduce a finite partition of unity on the real line given by m+2 nonnegative C^∞ functions $\sigma_j(\lambda)$ with $\sum_0^{m+1} \sigma_j(\lambda) \equiv 1$ and with λ_i outside the supports of all the σ_j but σ_i, $i = 1,\ldots,m$. The

supports of $\sigma_1,\ldots,\sigma_m$ are finite while the supports of σ_o, σ_{m+1} extend to $-\infty$ and $+\infty$ respectively. Now set

$$R_j(\lambda) = \sigma_j(\lambda)R(\lambda) , \quad w_j(\lambda) = R_j(\lambda)\hat{f}(\lambda) , \qquad j = 0,\ldots,m+1 ,$$

so that $\hat{v} = \sum w_j$.

Consider one of the R_j for $1 \leq j \leq m$. Since $R(\lambda)$ has a pole at λ_j of order $\leq k$ it admits the following expansion in an interval containing the support of σ_j

$$(3.5)' \qquad R(\lambda) = \sum_1^k (\lambda - \lambda_j)^{-r}P_r + P_o(\lambda)$$

where P_r, $r = 1,\ldots,k$ are bounded fixed operators, and $P_o(\lambda)$ is a regular operator valued function in the rectangle: $\{$Re λ in a slightly larger interval, and $-\varepsilon <$ Im $\lambda < a\}$. Using the Cauchy integral theorem we may infer that the derivatives $(\frac{d}{d\lambda})^r P_o(\lambda)$, $r \leq n+k$ have bounded $|\ \ |_X$ norms on the support of σ_j. Now

$$(3.6) \qquad (\lambda-\lambda_j)^k R_j(\lambda) = \sum_{r=1}^k (\lambda-\lambda_j)^{k-r} \sigma_j(\lambda)P_r + \sigma_j(\lambda)P_o(\lambda)(\lambda-\lambda_j)^k .$$

Near λ_j, $\hat{v} = w_j$, and hence w_j belongs to L_2. Since $w_j(\lambda)$ has compact support it is the Fourier transform of an analytic function $v_j(t)$ belonging to L_2.

We wish to estimate the norm of $v_j(t)$. According to (3.6) we have

$$(3.6)' \qquad (\lambda-\lambda_j)^k w_j(\lambda) = \sum_{r=1}^k (\lambda-\lambda_j)^{k-r} \sigma_j(\lambda)P_r\hat{f}(\lambda)$$
$$+ \sigma_j(\lambda)(\lambda-\lambda_j)^k P_o(\lambda)\hat{f}(\lambda) .$$

Differentiating this and using the fact that the derivatives of $P_o(\lambda)$ and of $\sigma_j(\lambda)$ are bounded we find

$$(3.6)'' \quad |(\tfrac{d}{d\lambda})^{n_o}(\lambda-\lambda_j)^k w_j(\lambda)|_X \leq C_1 \sum_{r=0}^{n_o} |(\tfrac{d}{d\lambda})^r \hat{f}(\lambda)|_Y , \quad n_o \leq n+k .$$

Since the inverse Fourier transform of $(\frac{d}{d\lambda})^{n_o}(\lambda-\lambda_j)^k w_j(\lambda)$, being in L_2, is

$$(\tfrac{t}{i})^{n_o}(D-\lambda_j)^k v_j(t) = (\tfrac{t}{i})^{n_o} e^{i\lambda_j t} D^k(e^{-i\lambda_j t} v_j(t)) ,$$

it follows by Plancherel's theorem that

$$|(1+t^{n+k})|D^k(e^{-i\lambda_j t} v_j(t))|_X|_{L_2} \leq C_2|(1+t^{n+k})|f(t)|_Y|_{L_2} .$$

We can now apply a well-known inequality of Hardy (see Hardy, Littlewood, Polya [1], Theorem 330) according to which, for scalar functions $a(t)$,

$$\int_0^\infty |t^n a(t)|^p \, dt \leq \text{constant} \int_0^\infty |t^{n+k} D^k a(t)|^p \, dt , \qquad p \geq 1$$

provided the right-hand side is finite, and $a(t) \in L_p$. (This is proved by repeated integrations by parts.) Since the derivative of a norm of a vector valued function of t is not greater than the norm of the derivative we deduce that (here L_2 norm represents the norm on $t > 0$)

$$(3.7) \quad |(1+t^n)|v_j(t)|_X|_{L_2} = |(1+t^n)|e^{-i\lambda_j t} v_j(t)|_X|_{L_2}$$

$$\leq C_3|(1+t^{n+k})|f|_Y|_{L_2} , \quad j = 1,\dots,m$$

and also that

$$(3.8) \qquad |(1+t^{n+1})|(D-\lambda_j)v_j(t)|_X|_{L_2}$$

$$\leq C_3|(1+t^{n+k})|f|_Y|_{L_2}, \qquad j = 1,\ldots,m .$$

Consider finally the functions $w_j(\lambda) = R_j(\lambda)\hat{f}(\lambda)$ for $j = 0$ and $m+1$. Since $|R(\lambda)|_X$ is uniformly bounded (away from the poles) we infer as before with the aid of the Cauchy integral formula that $R_o(\lambda)$ and $R_{m+1}(\lambda)$ have derivatives up to order n with uniformly bounded $|\ |_X$ norms. Therefore we see as above, that w_o and w_{m+1} are Fourier transforms of L_2 functions $v_o(t)$ and $v_{m+1}(t)$ with

$$(3.9) \quad |(1+|t|^n)|v_j(t)|_X|_{L_2} \leq C_4|(1+t^n)|f(t)|_Y|_{L_2}, \quad j=0 \text{ and } m+1.$$

If we now combine inequalities (3.7), (3.9) and the identity $v(t) = \sum_0^{m+1} v_j(t)$, we obtain (3.3). This completes the proof of (a).

(b) The proof of (3.4) is very similar. Using the Cauchy integral formula again we infer now that the derivatives up to order n of $\lambda R_o(\lambda)$ and $\lambda R_{m+1}(\lambda)$ are bounded in norm $|\ |_Y$, as $|\lambda| \to \infty$. Thus for $j = 0$ and $m+1$

$$|(\tfrac{d}{d\lambda})^{n_o}\lambda w_j(\lambda)|_Y \leq C_1' \sum_{r=0}^{n_o} |(\tfrac{d}{d\lambda})^r\hat{f}(\lambda)|_Y$$

so that

$$(3.9)' \quad |(1+t^{n_o})|Dv_j(t)|_Y|_{L_2} \leq C_2'|(1+t^{n_o})|f|_Y|_{L_2}, \quad n_o \leq n+1 .$$

For $1 \leq j \leq m$ we obtain from (3.8), with n replaced by n-1, the inequality

$$|(1+t^n)|Dv_j|_X|_{L_2} \leq |\lambda_j||(1+t^n)|v_j|_X|_{L_2} + C_3|(1+t^{n+k-1})|f|_Y|_{L_2}$$
$$\leq C_3'|(1+t^{n+k})|f|_Y|_{L_2}$$

by (3.7). Combining these inequalities we obtain (3.4).

If there are no poles then we may take $\sigma_j = 0$, $j = 1,\ldots,m+1$ and then (3.4)' follows from (3.9). If $\lambda = 0$ is the only pole, (3.4)' is derived by combining (3.8) and (3.9)'. Q.E.D.

By slight modifications of the preceding proofs one easily verifies the following.

Remarks 1) Suppose we are given operators P_j and Q_j as in §2, and suppose that the operators $P_{jS}(\lambda)$, $Q_{jS}(\lambda)$ are regular in the strip $-\varepsilon < \operatorname{Im}\lambda < a$, $\varepsilon, a > 0$ with the possible exception of a finite number of poles on the real axis of maximal order k and suppose that (see (2.8)) $K(\lambda) = O(1)$ for $|\lambda| \longrightarrow \infty$ in the strip. Let u(t) be a solution with $\|u(t)\| \in L_2$ (see (2.6)') on $t > 0$ of

$$|Lu|_Y \leq \frac{c}{(1+t)^k}\|u\| + b(t)$$

with b(t) as in Theorem 1.2. Then the same conclusions of the theorem hold, with $|u|_X$ replaced by $\|u\|$. If furthermore $|R_S(\lambda)|_Y = O(\frac{1}{\lambda})$ for $|\lambda| \longrightarrow \infty$ in the strip then the conclusions of Theorem 1.2' hold if u satisfies (3.1)' or (3.1)", with $|u|_X$ replaced by $\|u\|$.

2) Suppose $S = Y$ and suppose $R_S(\lambda) = R_Y(\lambda)$ satisfies the conditions of Theorem 1.2'. If $u(t)$ is a solution of $Lu = 0$ with $|u|_X \in L_2$ on $t > 0$ then derivatives $D^k u$ of all orders exist, are strongly continuous in Y, for $t > 0$ and their $|\ |_Y$ norms decay exponentially. This is a consequence of Corollary 1 of Theorem 2.2 of §5 (applied with $X = Y$), for if $|R_Y(\lambda)|_Y = O(\frac{1}{\lambda})$ on the real axis for $|\lambda| \longrightarrow \infty$ it follows that there are constants c, C such that $R_Y(\lambda)$ is regular in the region $|\mathrm{Re}\ \lambda| \geq c$, $|\mathrm{Im}\ \lambda| < C|\mathrm{Re}\ \lambda|$, and $|R_Y(\lambda)|_Y = O(\frac{1}{\lambda})$ as $|\lambda| \longrightarrow \infty$ in this region.

In Chapter 4 we prove in fact that the solution $u(t)$ as an element of Y is analytic in t.

3) Theorem 1.2 and Lemma 1.2 (a) hold under the following weaker hypotheses on the resolvent R_S:

(i) $R_S(\lambda)$ is regular in a strip $-\varepsilon < \mathrm{Im}\ \lambda < a$ except possibly for an <u>infinite</u> number of poles on the real axis, of maximal order k, such that the distance between any two of them is greater than a fixed positive number d.

(ii) There are positive constants M, d' such that $|R_S(\lambda)|_X \leq M$ for every λ in the strip whose distance to the set of real poles exceeds d'.

Since the theorem follows from Lemma 1.2 (a) we shall only indicate the necessary modifications in the proof of the latter. There are two cases to be considered: the poles are either bounded to one side or extend to infinity in both directions. We shall consider merely the first case supposing, say, that the poles go $+\infty$ and so (being clearly denumerable) may be enumerated as an increasing sequence λ_j, $j = 1,2,\ldots$.

As in the proof of Lemma 1.2 we introduce a partition of unity on the real line given by C^∞ functions $\sigma_j(\lambda)$, $j = 0,1,\ldots,$ with $\sum_0^\infty \sigma_j(\lambda) \equiv 1$ such that (a) any point on the real axis is contained in the supports of at most two of the σ_j, (b) the supports of σ_j for $j > 0$ is compact while that of σ_o extends to $-\infty$, (c) each λ_i is outside the supports of all σ_j but σ_i, (d) the functions σ_j and their derivatives up to order $(n+k)$ are bounded in absolute value by a constant K (this is possible because $|\lambda_i - \lambda_j| > d$ for $i \neq j$). Set

$$R_j(\lambda) = \sigma_j(\lambda)R_S(\lambda) \ , \quad w_j(\lambda) = R_j(\lambda)\hat{f}(\lambda) \ , \qquad j = 0,1,\ldots \ ,$$

and let v_j denote the inverse Fourier transform of $w_j(\lambda)$. For any fixed $j > 0$ we find, using condition (ii), and the Cauchy integral theorem, that (3.5)', (3.6) and (3.6)' hold, with the derivatives of $P_o(\lambda)$ up to order $n+k$ having $|\ |_X$ norm bounded by a fixed constant (independent of j) on the support s_j of σ_j, so that (3.6)" holds. It follows then as in the lemma that

$$\int_{-\infty}^{\infty} |t^{n_o}D^k(e^{-i\lambda_j t}v_j)|_X^2 \, dt \leq C_2 \sum_{r=0}^{n_o} \int_{s_j} |(\tfrac{d}{d\lambda})^r \hat{f}(\lambda)|_Y^2 \, d\lambda \ , \quad n_o \leq n+k$$

(C_2 independent of j). Applying again the inequality of Hardy we find that

$$\int |(\tfrac{d}{d\lambda})^{n_o} w_j(\lambda)|_X^2 \, d\lambda = \int_{-\infty}^{\infty} |t^{n_o}v_j|_X^2 \, dt$$

$$\leq C_3 \sum_0^{n+k} \int_{s_j} |(\tfrac{d}{d\lambda})^r \hat{f}(\lambda)|_X^2 \, d\lambda \ , \qquad n_o \leq n \ .$$

Similarly for $j = 0$ we find

$$\int |(\tfrac{d}{d\lambda})^{n_0} w_0(\lambda)|_X^2 \, d\lambda \leq C_3 \sum_0^{n_0} \int_{s_0} |(\tfrac{d}{d\lambda})^r \hat{f}(\lambda)|_X^2 \, d\lambda \,, \qquad n_0 \leq n \,.$$

Since $\hat{v} = \sum_0^\infty w_j(\lambda)$ and since each point λ lies in at most two of the sets s_j it follows that for $n_0 \leq n$

$$\int_0^\infty |t^{n_0} v|_X^2 \, dt = \int_{-\infty}^\infty |(\tfrac{d}{d\lambda})^{n_0} \hat{v}(\lambda)|_X^2 \, d\lambda$$

$$\leq 2 \sum_{j=0}^\infty \int |(\tfrac{d}{d\lambda})^{n_0} w_j(\lambda)|_X^2 \, d\lambda \leq 4C_3 \sum_0^{n+k} \int_{-\infty}^\infty |(\tfrac{d}{d\lambda})^r \hat{f}(\lambda)|_Y^2 \, d\lambda$$

$$\leq C_4 \int_0^\infty |(1 + t^{n+k}) f(t)|_Y^2 \, dt \,,$$

which is the desired inequality (3.3).

The hypotheses in Remark 1 on the operators P_{jS}, Q_{jS} may be weakened in a similar manner.

We shall give an extension to L_p, $1 < p < \infty$, of Lemma 1.2 (b) and Theorem 1.2' based on a "Multiplier Theorem" of Michlin [1]. A new proof of the theorem was recently given by Hörmander [1]; J. T. Schwartz [1] has observed that the results of Chapter 2 in Hörmander's paper are valid also for vector valued functions. In particular, the following form of the theorem holds.

Multiplier Theorem: Let $T(\lambda)$ be a C^1 operator valued function defined on the real line whose value for every λ is a bounded operator mapping some Hilbert space S into another Hilbert space. Assume that there is a constant K such that $T(\lambda)$ and $\lambda \frac{dT}{d\lambda}$ are

bounded in norm by K. Then the operation $\tilde{T}$ - on functions s(t), $-\infty < t < \infty$ with values in S, such that $||s(t)|_S|_{L_p} < \infty$, - defined as follows: operate on the Fourier transform $\hat{s}(\lambda)$ of s(t) by $T(\lambda)$ and take the inverse Fourier transform, is a bounded operator in L_p, i.e.

$$||\tilde{T}s|_X|_{L_p} \leq \text{constant}\cdot K||s(t)|_S|_{L_p}$$

where the constant depends only on p.

Now the generalization of Lemma 1.2 (b).

Lemma 1.2': Let Q be a linear operator mapping the domain of A into X and assume that $QR_S(\lambda) = Q_S(\lambda)$ is regular for all λ in a strip $-\varepsilon < \operatorname{Im}\lambda < a$, for $\varepsilon, a > 0$, except possibly for a finite number of real poles $\lambda_1,\ldots,\lambda_m$ of maximal order k, and assume that

$$|Q_S(\lambda)|_X + |\lambda \frac{d}{d\lambda} Q_S(\lambda)|_X = O(1) \quad \text{for } |\lambda| \to \infty \text{ in the strip.}$$

Let $p > 1$ be finite. For every nonnegative integer n there exists a constant c_n depending only on A, Q, n, p such that for any function v(t) vanishing at the origin with $|Qv|_X$, $(1+t^{n+k})|Lv|_Y$ in L_p on $t > 0$ the inequality

$$|(1+t^n)|Qv|_X|_{L_p} \leq c_n|(1+t^{n+k})|Lv|_Y|_{L_p}$$

holds. The same holds with X replaced by Y everywhere.

We shall merely sketch the proof. Taking Fourier transforms as in the proof of Lemma 1.2 (one should first carry this out for

functions $v(t)$ with compact support, and then reduce the general case to this one with the aid of the ζ operator) we obtain the analogous decomposition

$$Qv = \sum_0^{m+1} v_j(t) \ , \qquad \hat{v}_j = w_j(\lambda) = \sigma_j(\lambda)QR(\lambda)\hat{f}(\lambda) \ .$$

As in the proof of Lemma 1.2 we find that for $1 \leq j \leq m$, $n_o \leq n+k$, the term $(\frac{d}{d\lambda})^{n_o}(\lambda-\lambda_j)^k w_j(\lambda)$ is given by a sum of bounded operators (having also bounded derivatives) acting on terms $(\frac{d}{d\lambda})^r\hat{f}(\lambda)$ for $r \leq n_o$. We may therefore apply the Multiplier Theorem and conclude that

$$|(1+t^{n+k})|D^k(e^{-i\lambda_j t}v_j(t))|_X|_{L_p} \leq C_2'|(1+t^{n+k})|f(t)|_Y|_{L_p} \ .$$

Using the inequality of Hardy again we find that

$$(3.8)' \quad \int_0^\infty |(1+t^n)|v_j|_X|^p \, dt \leq C_3' \int_0^\infty |(1+t^{n+k})|f(t)|_Y|^p \, dt \ .$$

We have still to consider w_o and w_{m+1}; consider just $w_o = \sigma_o(\lambda)QR\hat{f}$. Because of our hypothesis on $QR(\lambda)$ we find, using the Cauchy integral theorem, that the $|\ |_X$ norms of derivatives of $\sigma_o(\lambda)QR(\lambda)$ are $O(\frac{1}{\lambda})$ as $|\lambda| \longrightarrow \infty$ on the real axis. We conclude, as above, with the aid of the Multiplier Theorem, that

$$\int_0^\infty |(1+t^{n_o})|v_o(t)|_X \, dt \leq C_4' \int_0^\infty |(1+t^{n_o})|f|_Y|^p \, dt \ , \qquad n_o \leq n+1 \ .$$

The remainder of the proof is similar to that of Lemma 1.2.

With the aid of Lemma 1.2' one may now prove a generalization of Remark 1 and of Theorem 1.2', which we shall use in the applications. We observe that if the hypotheses of Remark 1 hold, with c small, and with one of the Q_j equal to A, then also (3.1)' holds. In the following p, $1 < p < \infty$, is fixed.

Theorem 1.2": Let P_j, Q_j be a finite number of operators as in §2, and set $\|u\| = \sum (|P_j u|_X + |Q_j u|_Y)$. Assume that $P_j R_S(\lambda) = P_{jS}$, $Q_j R_S(\lambda) = Q_{jS}$ are regular in a strip $-\varepsilon < \operatorname{Im} \lambda < a$, with $\varepsilon, a > 0$, except possibly for a finite number of poles $\lambda_1, \ldots, \lambda_m$ on the real axis of maximal order k. Assume also that

$$|P_{jS}(\lambda)|_X, \quad |\lambda \tfrac{d}{d\lambda} P_{jS}(\lambda)|_X, \quad |Q_{jS}(\lambda)|_Y, \quad |\lambda \tfrac{d}{d\lambda} Q_{jS}(\lambda)|_Y \ \text{are } O(1)$$

as $|\lambda| \longrightarrow \infty$ in the strip. Let $b(t)$ be a scalar function on $t > 0$ such that for some positive $a' < a$, $e^{a't}b(t)$ belongs to L_p, or $(1+t)^k b \in L_p$. Then there exists a positive number c" depending only on A, the P_j and Q_j, p (and κ) such that if $u(t)$ is a solution with $|P_j u|_X$, $|Q_j u|_Y$, $|Lu|_Y \in L_p$ for $t > 0$ of

$$|Lu(t)|_Y \leq \frac{c}{(1+t)^k} \|u\| + b(t)$$

with $c \leq c''$ then $u(t)$ satisfies

$$\int_0^\infty |e^{a't}\|u\||^p dt \leq C \int_0^1 \|u\|^p \, dt + C \int_0^\infty |e^{a't}b|^p \, dt ,$$

$$\int_0^\infty \|u\|^p \, dt \leq C \int_0^1 \|u\|^p \, dt + C \int_0^\infty |(1+t^k)b(t)|^p \, dt$$

with C some constant.

We mention that if $S = Y$, and if $|\frac{d}{d\lambda} R_Y(\lambda)|_X = O(\frac{1}{\lambda})$ for $|\lambda| \longrightarrow \infty$ on the real axis, then $R_Y(\lambda)$ exists in a larger region including the set $\{\lambda \,|\, |\mathrm{Im}\ \lambda| < \text{constant } |\mathrm{Re}\ \lambda|^{1/2}, |\mathrm{Re}\ \lambda| > \text{constant}\}$. In this set $|R_Y(\lambda)|_X$ is bounded. Furthermore if $|R_Y(\lambda)|_X = O(\lambda^{-1/2})$ for $|\lambda| \longrightarrow \infty$ on the real axis then $|\frac{d}{d\lambda} R_Y(\lambda)|_X = O(\lambda^{-1})$ as $|\lambda| \longrightarrow \infty$ on the real axis.

Chapter II

Asymptotic Expansions and Completeness

4. Asymptotic expansions in Hilbert space

In this section we assume that X and Y are Hilbert spaces and consider solutions of $Lu = 0$ on $t > 0$, for which we can obtain much more precise asymptotic results.

Theorem 2.1: Let $u(t)$ be a solution of $Lu = 0$ with $|u|_X \in L_2$ on $t > 0$. Assume that $R_S(\lambda)$ is meromorphic in a strip $0 \leq \operatorname{Im} \lambda < a$ and that $|R_S(\lambda)|_X = O(1)$ as $|\lambda| \to \infty$ in the strip. Then for every $\varepsilon > 0$ there is a finite sum of exponential solutions $u_j(t) = e^{i\lambda_j t} p_j(t)$, $j = 1,\ldots,m$ of $Lu = 0$ where λ_j are the poles of $R_S(\lambda)$ in the strip $0 < \operatorname{Im} \lambda \leq a-\varepsilon$ such that

$$|e^{(a-\varepsilon)t}|u(t) - \sum_1^m u_j(t)|_X|^2_{L_2} \leq \text{constant} \int_0^1 |u(t)|^2_Y \, dt$$

where the constant depends only on A, a and ε.

Proof: Making use of the $\zeta\cdot$ operator of §2 we set $v = \zeta\cdot u$, $f = Lv$; then $f(t) = 0$ for $t > 1$ and $||f|_Y|^2_{L_2} \leq \text{constant} \int_0^1 |u|^2_Y \, dt$. The Fourier transforms $\hat{v}(\lambda)$, $\hat{f}(\lambda)$ of v and f are related by $\hat{v}(\lambda) = R_S(\lambda)\hat{f}(\lambda)$ on the resolvent set ρ_S. Since f has compact support $\hat{f}$ is an entire function of exponential type one (the L_2 norm of $|\hat{f}|_Y$ on a line $\operatorname{Im} \lambda = \sigma > 0$ is not greater than e^{σ} times its L_2 norm on the real axis); $\hat{v}(\lambda)$ is defined as an analytic function in the lower half plane with values in L_2 on the real axis. Thus the relation $\hat{v}(\lambda) = R_S(\lambda)\hat{f}(\lambda)$ enables us to extend

$\hat{v}(\lambda)$ as a meromorphic function into the region $\mathrm{Im}\,\lambda < a$. Since $|\hat{v}|_X \in L_2$ on the real axis $\hat{v}(\lambda)$ has no poles there.

Let δ, $a-\varepsilon < \delta < a$, be such that the strip $0 \leq \mathrm{Im}\,\lambda \leq \delta$ contains no poles of $R_S(\lambda)$ besides those in the strip $0 \leq \mathrm{Im}\,\lambda \leq a-\varepsilon$. Then the integral

$$w(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathrm{Im}\,\lambda=\delta} e^{i\lambda t} R_S(\lambda)\hat{f}(\lambda)d\lambda$$

integrated from left to right differs from the corresponding integral along the real axis by $i\sqrt{2\pi}$ times the residues of $e^{i\lambda t}R_S(\lambda)\hat{f}(\lambda)$ in the strip $0 < \mathrm{Im}\,\lambda < a-\delta$. This is proved by applying the Cauchy formula to a rectangle with corners at $-N$, N, $N+i(a-\delta)$, $-N+i(a-\delta)$ and letting $N \to \infty$. Since $\hat{f}(\lambda) \in L_2$ on the real axis and is of exponential type it is easily seen that the contributions of the vertical sides of the rectangle go to zero as $N \to \infty$. It is also easily seen that the residues are exponential solutions of $Lu = 0$.

Thus we see that $v(t)$ differs from the sum of exponential solutions as described by $w(t)$. By Plancherel's theorem it follows that

$$\left| e^{\delta t} |w(t)|_X \right|_{L_2}^2 = \int_{\mathrm{Im}\,\lambda=\delta} |R_S(\lambda)\hat{f}(\lambda)|_X^2 \, d\lambda \leq \text{constant} \int_{i\delta-\infty}^{i\delta+\infty} |\hat{f}(\lambda)|_Y^2 \, d\lambda$$

$$\leq \text{constant} \int_{-\infty}^{\infty} |\hat{f}(\lambda)|_Y^2 \, d\lambda \,,$$

since f is of exponential type and $|R_S(\lambda)|_X$ is bounded on $\mathrm{Im}\,\lambda = \delta$,

$$= \text{constant} \; \left|\,|f(t)|_Y\right|_{L_2}^2 \,,$$

and the desired result follows easily.

From the form of the result it is clear that the exponential solutions u_j so obtained are independent of the particular $\zeta\cdot$ operator used. They depend only on $u(t)$.

Corollary 1: Assume that $R_S(\lambda)$ is actually meromorphic in a larger strip $-\varepsilon < \operatorname{Im}\lambda < a$ and satisfies $|R_S(\lambda)|_X = O(1)$ as $|\lambda| \longrightarrow \infty$ in this strip (if $S = Y$ this follows from our hypothesis). The result of the theorem then holds if in place of the assumption $|u|_X \in L_2$ for $t > 0$ we assume $|u|_X$ belongs to L_p, for some $p \geq 1$, or to L^0_∞ for $t > 0$. (By L^0_∞ we mean L_∞ functions tending to zero at infinity.)

This is easily derived from the theorem. For ε sufficiently small the strip $-\varepsilon < \operatorname{Im}\lambda < a$ will contain no new poles of $R_S(\lambda)$. Set $u_\varepsilon(t) = e^{-\varepsilon t/2}u(t)$. Then $u_\varepsilon(t)$ belongs to L_2 for $t > 0$ and satisfies $(L - i\frac{\varepsilon}{2})u_\varepsilon(t) = 0$. Applying the theorem we find that

$$\left| e^{(a-\varepsilon)t} \left| u(t) - \sum e^{i\lambda_j t} p_j(t) - \sum e^{i\lambda_k^* t} p_k^*(t) \right|_X \right|_{L_2} \leq \text{constant} \int_0^1 |u(t)|_Y^2 \, dt$$

where λ_j are the poles of $R_S(\lambda)$ in the strip $0 < \operatorname{Im}\lambda \leq a-\varepsilon$, while λ_k^* are the poles of $R_S(\lambda)$ on the <u>real</u> axis.

Since however $|u(t)|_X$ belongs to L_p or L^0_∞ it follows that the $p_k^*(t)$ must vanish. Q.E.D.

By the same argument one obtains

Corollary 2: Under the same hypotheses on $R_S(\lambda)$ as in the theorem, assume that $u(t)$ is a solution of $Lu(t) = 0$ in $t \geq 0$ such that $(1+t)^N |u(t)|_X$ belongs to L_p for some $p \geq 1$ or to L_∞; here N is a positive number. Then the same conclusion holds as in the theorem, except that the λ_j are now the poles in the closed strip $0 \leq \operatorname{Im} \lambda \leq a-\varepsilon$ of $R_S(\lambda)$.

This device of considering $e^{-\sigma t}u(t)$ in place of $u(t)$ may be used to derive analogous results for solutions of $Lu = 0$ which are allowed to increase exponentially, i.e. such that $e^{\sigma t}u(t)$ belongs to L_p. One then assumes that $R_S(\lambda)$ is meromorphic in some strip $-\sigma \leq \operatorname{Im} \lambda < a$.

Remark 1: Consider again a solution u of $Lu = 0$ with $|u|_X \in L_2$ on $t > 0$. Let P be a closed operator with domain in X containing the domain of A and with range in X (one might also consider the case where the range of P is in Y), such that $|Pu(t)|_X \in L_2$. Suppose that $PR(\lambda)$ is meromorphic in the strip $0 \leq \operatorname{Im} \lambda < a$ and satisfies $|PR(\lambda)|_X = O(1)$ as $|\lambda| \longrightarrow \infty$ in the strip. If $\lambda_1, \ldots, \lambda_m$ are the poles of $PR(\lambda)$ in $0 < \operatorname{Im} \lambda \leq a-\varepsilon$ then there is a finite number of exponential polynomials $v_j(t) = e^{i\lambda_j t} q_j(t)$, q_j are polynomials, such that

$$\left| e^{(a-\varepsilon)t} \left| Pu(t) - \sum_1^m v_j(t) \right|_X \right|_{L_2}^2 \leq \text{constant} \int_0^1 |u|_Y^2 \, dt \, .$$

In case the λ_j are also poles of $R(\lambda)$ then we can assert that each $v_j(t) = Pu_j(t)$ where $u_j = e^{i\lambda_j t} p_j(t)$ is an exponential solution of $Lu = 0$.

The proof of the remark is similar to that of the theorem. As before we set $v = \zeta \cdot u$, $f = Lv$, and take Fourier transforms. It is easily seen that for $\mathrm{Im}\,\lambda < 0$, $P\hat{u}(\lambda)$ is the Fourier transform of Pu, hence this holds also for λ real. The formula $\widehat{Pu} = PR(\lambda)\hat{f}$ therefore holds almost everywhere on the real axis, and enables us to extend $\widehat{Pu}$ as a meromorphic function into the strip. The rest of the proof is as before.

Similar remarks apply to the corollaries.

Assume now that $R_S(\lambda)$ is meromorphic in the half plane $\mathrm{Im}\,\lambda \geq 0$ (we may also consider just $PR(\lambda)$ to be meromorphic there). With any solution u of $Lu = 0$, with $|u|_X \in L_2$ on $t > 0$ we shall associate a formal "Fourier expansion" of exponential solutions in the following way. The Fourier transform $\hat{v}$ of the function $v = \zeta \cdot u$, $\hat{v}(\lambda) = R_S(\lambda)\hat{f}(\lambda)$, is now meromorphic in the whole upper half plane $\mathrm{Im}\,\lambda > 0$. If it has no poles its "Fourier expansion" will be zero, and we write $u \sim 0$. If it has poles (which are, of course, also poles of $R_S(\lambda)$) we associate with each λ_j the exponential solution $u_j = e^{i\lambda_j t} p_j(t)$ given by $\sqrt{2\pi i}$. Residue at λ_j of $(e^{i\lambda t}R_S(\lambda)\hat{f}(\lambda))$. It is convenient to arrange these poles first according to increasing values of $\mathrm{Im}\,\lambda$ and then according to decreasing order of the polynomial $p_j(t)$. With this convention $\sum u_j(t)$ will be the formal "Fourier expansion" of u in exponential solutions. Applying Theorem 2.1 we obtain the following

<u>Theorem 2.1$'$</u>: <u>Suppose that</u> $R_S(\lambda)$ <u>is meromorphic in</u> $\mathrm{Im}\,\lambda \geq 0$ <u>and satisfies in every strip</u> $0 \leq \mathrm{Im}\,\lambda < a$, $|R_S(\lambda)|_X = O(1)$ <u>as</u> $|\lambda| \to \infty$ <u>in the strip</u>. <u>Let</u> $u(t)$ <u>be a solution of</u> $Lu = 0$ <u>in</u> $t > 0$ <u>with</u> $|u|_X$

square integrable. Then the Fourier expansion of u is an asymptotic expansion

$$u(t) \sim \sum u_j(t)$$

in the sense that if $u_k(t)$ is an exponential solution of index m_k (one less than the order of the associated polynomial corresponding to the eigenvalue λ_k) then for any $\varepsilon > 0$

$$\left|(1+t)^{\frac{1}{2}-m_k-\varepsilon}\, e^{\operatorname{Im}\lambda_k t}\left|u(t) - \sum_{j=1}^{k-1} u_j(t)\right|_X\right|_{L_2} < \infty .$$

Clearly similar asymptotic expansions for Pu as a sum of exponential polynomials may be obtained under analogous hypotheses on $PR(\lambda)$. Sufficient conditions for the exponential solutions to be complete (in some sense) in the class of solutions will be given in §7.

In establishing our Phragmén-Lindelöf estimate, Theorem 2.1, we have permitted $R(\lambda)$ to have a finite number of poles on the real axis. Actually we may allow much worse singularities on the axis and still obtain our result. To illustrate this we prove the following, where we assume $S = Y$ and write $R_S(\lambda) = R(\lambda)$.

Theorem 2.1": Assume that $R(\lambda)$ is regular at every point in a strip $0 \leq \operatorname{Im}\lambda < a$ with the exception of a denumerable number of points λ_j on the axis. Assume also that in the interior $|R(\lambda)|_X \leq M(\operatorname{Im}\lambda)^{-N}$ for some constants M, N. If $u(t)$ is a solution of $Lu = 0$ on $t > 0$, with $|u|_X \in L_2$ then for any $\varepsilon > 0$ we have

$$(4.1) \qquad |e^{(a-\varepsilon)t}|u|_X|_{L_2} \le \underline{\text{constant}} \int_0^1 |u(t)|_Y^2 \, dt \ ,$$

<u>where the constant depends only on</u> A, a <u>and</u> ε.

<u>Proof</u>: Let σ be a C^∞ function which vanishes for $t \le 0$, is equal to one for $t \ge 1$ and is monotonic increasing. Set $v = \sigma u$, $f = Lv$ and take Fourier transforms. The Fourier transforms satisfy

$$\hat{v}(\lambda) = R(\lambda)\hat{f}(\lambda)$$

for λ in the resolvent set. This enables us to extend $\hat{v}(\lambda)$, which is analytic in $\operatorname{Im} \lambda < 0$ and belongs to L_2 on the real axis, as an analytic function in the region $\operatorname{Im} \lambda < a$, with the exception of the points λ_j.

Since $|\hat{v}(\lambda)|_X \in L_2$ on the real axis one finds easily, with the aid of Cauchy's integral formula that

$$(4.2) \qquad |\hat{v}(\lambda)|_X \le \text{constant } |\operatorname{Im} \lambda|^{-1/2} \ , \qquad \operatorname{Im} \lambda < 0 \ .$$

We wish to show that $\hat{v}(\lambda)$ is analytic at every point on the real axis. Since the set of points on the axis where $\hat{v}(\lambda)$ is not analytic is closed and denumerable it has an isolated point, unless it is empty. Assume then that λ_1 is an isolated point of this set; we may suppose that λ_1 is the origin. Thus $\hat{v}(\lambda)$ is analytic in a circle $|\lambda| < \delta$ except at the origin, and $|\hat{v}(\lambda)|_X \le$ some constant K_1 on $|\lambda| = \delta$. Furthermore, since $|\hat{f}(\lambda)|_Y$ is bounded for $\operatorname{Im} \lambda < a$, we have by our hypothesis on $R(\lambda)$ and by (4.2) (we may suppose $N \ge 1/2$)

$$(4.3) \qquad |\hat{v}(\lambda)|_X \le K_2|\text{Im } \lambda|^{-N} \qquad \text{in} \quad |\lambda| \le \delta ,$$

for some constant K_2.

If we consider the region $|\lambda| \le \delta$, Re $\lambda > \varepsilon$, for $0 < \varepsilon < \delta$, we see that on its boundary $|(\lambda-\varepsilon)^N\hat{v}(\lambda)|_X \le K_3$, where K_3 is a fixed constant independent of ε. It follows from the maximum principle that in this region $|\hat{v}(\lambda)|_X \le K_3|\lambda-\varepsilon|^{-N}$. Letting $\varepsilon \to 0$ we find that $|\hat{v}(\lambda)|_X \le K_3|\lambda|^{-N}$ for Re $\lambda > 0$. A similar estimate holds for Re $\lambda < 0$, and we may conclude that $|\hat{v}(\lambda)|_X \le K_3|\lambda|^{-N}$ in the whole circle. But then $\hat{v}(\lambda)$ has a pole of order $\le N$ at the origin, and since $|\hat{v}(\lambda)|_X$ belongs to L_2 on the real axis $\hat{v}(\lambda)$ must in fact be regular at the origin. Thus $\hat{v}(\lambda)$ is analytic at every point on the real axis.

We have shown that $\hat{v}(\lambda) = R(\lambda)\hat{f}(\lambda)$ is regular for Im $\lambda < a$. On the line Im $\lambda = a-\varepsilon$ we have $|R(\lambda)|_X \le M(a-\varepsilon)^{-N}$. Furthermore the L_2 norm of $|\hat{f}(\lambda)|_Y$ on this line is not greater than $e^{a-\varepsilon}$ times the L_2 norm on the real axis. It follows that the L_2 norm of $|\hat{v}(\lambda)|_X$ on the line Im $\lambda = a-\varepsilon$ is bounded by $M(a-\varepsilon)^{-N}e^{a-\varepsilon}$ times the L_2 norm of $|f(t)|_Y$. But a standard argument shows that the L_2 norm of $|\hat{v}(\lambda)|_X$ on the line Im $\lambda = a-\varepsilon$ is equal to the L_2 norm of $e^{(a-\varepsilon)t}|v(t)|_X$, and the proof of (4.1) is complete.

We shall not make use of Theorem 2.1" and in the remainder of this chapter we will confine ourselves to cases where $R(\lambda)$ is meromorphic on the real axis.

5. Asymptotic expansions in Banach space

We continue our study of solutions of the homogeneous equation $Lu = 0$ on $t > 0$ assuming now that X and Y are Banach spaces and that $|u(t)|_X$ belongs to L_1. We shall establish analogues of the results in the preceding sections under the hypothesis that $R_S(\lambda)$ is regular in a much larger, striplike, region above the real axis, whose vertical width grows logarithmically. Since we shall permit the norm of $R_S(\lambda)$ to grow fairly rapidly at ∞ we shall not bother with the subspace S, i.e. we shall suppose that $S = Y$ and write $R_S(\lambda) = R(\lambda)$. Our estimates of Phragmén-Lindelöf type will now be pointwise estimates, i.e. $e^{-\varepsilon t}|u(t)|_X$ will be bounded, not merely some integral norm of it.

H. Tanabe [2] has considered differential equations (2) with $A = A(t)$ depending on t such that, for each t, $A(t)$ is the generator of a semigroup, and has proved, under certain conditions, that if $f(t) \to f(\infty)$ in Y and $A(t) \to A(\infty)$ in some sense then the solution $u(t)$ tends to a solution $u(\infty)$ of $A(\infty)u(\infty) = -f$. His proof makes use of the fundamental solution constructed in his paper [1].

For convenience we formulate

<u>Condition</u> $C_{\delta,a}$: $R(\lambda)$ is regular in a strip $\delta \leq \operatorname{Im}\lambda < a$ except for a finite number of poles, and for some constants c, C, $R(\lambda)$ is regular in the region

$$(5.1) \qquad |\operatorname{Re}\lambda| \geq c\ , \qquad \delta \leq \operatorname{Im}\lambda \leq C \log|\operatorname{Re}\lambda|\ ,$$

with $|R(\lambda)|_X = O(e^{\sigma|\lambda|})$ as $|\lambda| \longrightarrow \infty$ in the region, and $R(\lambda)$ satisfies $|R(\lambda)|_X = O(e^{\beta \operatorname{Im} \lambda})$ as $|\lambda| \longrightarrow \infty$ on the curved part of the boundary of the region (5.1). Here β, σ are nonnegative constants.

Theorem 2.2: Assume that $R(\lambda)$ satisfies Condition $C_{o,a}$ and let $\lambda_1,\ldots,\lambda_m$ be the poles of $R(\lambda)$ in the strip $0 < \operatorname{Im} \lambda < a$. Let $u_j(t) = e^{i\lambda_j t} p_j(t)$ be the residue of $e^{it\lambda}R(\lambda)u(0)$ at λ_j, $j = 1,\ldots,m$; the u_j are exponential solutions of $Lu = 0$. Let $a > a-\varepsilon > \operatorname{Im} \lambda_j$, $j = 1,\ldots,m$. Then for $t \geq \tau_j > \beta + \frac{1+j}{C}$ the solution $u(t)$ has strong derivatives (in X) with respect to t up to order j and

$$|D^j(u - \sum_1^m u_k)|_X \leq K_j e^{-t(a-\varepsilon)}|u(0)|_Y , \qquad t \geq \tau_j \ ; \quad j = 0,1,\ldots .$$

Here K_j is a constant depending only on the operator A and on τ_j.

This is related to Theorem 4.3.

Proof: We extend $u(t)$ to be zero for $t < 0$ and take its Fourier transform

$$\hat{u}(\lambda) = \frac{1}{\sqrt{2\pi}} \int e^{-i\lambda t}u(t)dt .$$

Since $|u(t)|_X \in L_1$, the transform $\hat{u}(\lambda)$ is regular analytic in the lower half plane $\operatorname{Im} \lambda < 0$ and continuous and bounded in its closure, and satisfies $(\lambda-A)\hat{u}(\lambda) = \frac{1}{i\sqrt{2\pi}} u(0)$, or

$$\hat{u}(\lambda) = \frac{1}{i\sqrt{2\pi}} R(\lambda)u(0) \qquad \text{for } \lambda \text{ in the resolvent set.}$$

This equation enables us to extend $\hat{u}(\lambda)$ as a meromorphic function to the union U of the strip $0 \leq \operatorname{Im} \lambda < a$, and the domain (5.1).

Since $\hat{u}(\lambda)$ is continuous on the real axis it does not have any poles there, hence the only possible poles it may have in U are $\lambda_1,\ldots,\lambda_m$.

Writing $\lambda = \xi + i\eta$, let Γ be an infinite arc lying in U composed of a line segment $\eta = a-\varepsilon$, $|\xi| < c'$, having the poles $\lambda_1,\ldots,\lambda_m$ beneath it; the two infinite arcs: $\eta = C \log |\frac{\xi}{c'}| + a-\varepsilon$ for $\xi \leq -c'$ and $\xi \geq c'$. We orient Γ according to increasing Re λ.

Now set

$$v(t) = \frac{1}{\sqrt{2\pi}} \int_\Gamma e^{it\lambda}\hat{u}(\lambda)d\lambda$$

the integral being absolutely convergent in X for $t > \beta + \frac{1}{C}$. Indeed on the infinite arcs of Γ we have

$$|e^{it\lambda}\hat{u}(\lambda)|_X \leq \text{constant } |\tfrac{\xi}{c'}|^{(\beta-t)C} e^{-t(a-\varepsilon)} |u(0)|_Y .$$

Thus for $t > \beta + \frac{1+j}{C}$, $D^j v$ exists and is given by the absolutely convergent integral

$$D^j v = \frac{1}{\sqrt{2\pi}} \int_\Gamma e^{it\lambda}\lambda^j\hat{u}(\lambda)d\lambda .$$

We also see that $v(t)$ and its derivatives are $O(e^{-t(a-\varepsilon)})$ as $t \longrightarrow \infty$. Let us verify this just for v itself. Clearly the contribution of the integral over the straight horizontal portion of Γ is $O(e^{-t(a-\varepsilon)})$, while the integral over the remaining infinite arcs has $|\ |_X$ norm bounded by

$$\text{constant } e^{-t(a-\varepsilon)} \int_{c'}^{\infty} |\tfrac{\xi}{c'}|^{(\beta-t)C} d\xi \leq \text{constant } e^{-t(a-\varepsilon)}$$

$$\text{for } (\beta-t)C < -1 .$$

Thus to conclude the proof of the theorem we have only to show that

$$u(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\lambda}\hat{u}(\lambda)d\lambda = v(t) + \sum_{1}^{m} u_j(t) .$$

This will follow on letting $k \to \infty$ Cauchy's integral formula applied to the region bounded on the sides by two vertical lines $\operatorname{Re}\lambda = \pm k$, k large, on top, by the portion of Γ with $|\operatorname{Re}\lambda| \le k$, and bounded on the bottom by the segment on the real axis: $|\operatorname{Re}\lambda| \le k$.

We must verify that the integrals on the vertical edges tend to zero as $k \to \infty$. To this end we shall make use of a Phragmén-Lindelöf argument to obtain a better estimate for $|\hat{u}(\lambda)|_X$ than the one given by our hypothesis on $R(\lambda)$:

$$|\hat{u}(\lambda)|_X = O(e^{\sigma|\lambda|}) \quad \text{as } |\lambda| \to \infty \text{ in } U .$$

Since $|u(t)|_X$ belongs to L_1 we know, in fact, by the Riemann-Lebesgue lemma, that $|\hat{u}(\lambda)|_X \to 0$ as $|\lambda| \to \infty$ on the real axis.

Consider $w(\lambda) = e^{i\beta'\lambda}\hat{u}(\lambda)$, with $\beta' > \beta$. Clearly then $|w(\lambda)|_X \to 0$ as $|\lambda| \to \infty$ on Γ or on the real axis. By the Phragmén-Lindelöf theorem (see Polya-Szegö [1], Section 3, problem 324) it follows that $|w(\lambda)|_X$ is bounded in U. Applying furthermore the result of problem 339 of Section 3 in the same book to $w(\lambda)$ we find that

$$|w(\lambda)|_X = o(1) \quad \text{as } |\lambda| \to \infty \text{ in } U.$$

We may now estimate the $|\ |_X$ norm of the integral on, say, the side $\operatorname{Re}\lambda = k$: for $t > \beta'$

$$\left| \int_{\mathrm{Re}\,\lambda=k} e^{it\lambda}\hat{u}(\lambda)d\lambda \right|_X \leq \int_0^{\infty} e^{(\beta'-t)\eta}\, d\eta \cdot o(1) = \frac{1}{t-\beta'}\, o(1) ,$$

which goes to zero as $k \longrightarrow \infty$. Q.E.D.

Following the proof of Corollary 1 of Theorem 2.1 we may prove

Corollary 1: Theorem 2.2 holds if, in place of the assumption $|u(t)|_X \in L_1$ for $t > 0$ we assume that $|u(t)|_X$ belongs to L_p, $p \geq 1$, or to L^0_∞ on $t > 0$, and if we also assume that $R(\lambda)$ satisfies condition $C_{\delta,a}$ for some $\delta < 0$.

It is clear that an analogue of Theorem 2.2' may also be stated, giving an asymptotic expansion for a solution of $Lu = 0$ with, say, $|u|_X \in L_p$ on $t > 0$, in case $R(\lambda)$ is meromorphic in $\mathrm{Im}\,\lambda \geq 0$ and satisfies condition $C_{\delta,a}$ for some $\delta < 0$, $a > 0$.

In analogy with Remark 1 after Theorem 2.1 we have

Remark 1: Let u be a solution of $Lu = 0$ on $t > 0$. Suppose we are given an operator P (as in that remark) such that $|u(t)|_X$ and $|Pu|_X$ belong to L_1 and such that $x(\lambda) = PR(\lambda)u(0)$ is regular in a strip $0 \leq \mathrm{Im}\,\lambda < a$ except for a finite number of poles, and regular in the region (5.1), and satisfies there $|x(\lambda)|_X = O(e^{\sigma|\lambda|})$ as $|\lambda| \longrightarrow \infty$ in the region and $|x(\lambda)|_X = O(e^{\beta\,\mathrm{Im}\,\lambda})$ as $|\lambda| \longrightarrow \infty$ on the curved boundary of the region. Then the conclusion of the theorem holds for Pu, where now u_k is the residue of $e^{it\lambda}PR(\lambda)u(0)$ at the pole λ_k. If λ_k is also a pole of $R(\lambda)u(0)$ then $u_k = Pv_k$ where v_k is an exponential solution of $Lu = 0$.

Similar remarks can be made in connection with Corollary 1, or for asymptotic expansions of Pu in terms of exponential polynomials.

In general, in deriving asymptotic expressions for a solution of $Lu = 0$ as a sum of exponentials, the larger the region in which $R(\lambda)$ is regular, the faster is the speed with which its norm may be permitted to grow at infinity. We illustrate this by deriving still another form of Theorems 2.1 and 2.2. Again we say $S = Y$ and denote $R_S(\lambda)$ by $R(\lambda)$.

Theorem 2.3: Assume that $R(\lambda)$ is regular in a strip $0 \leq \text{Im}\,\lambda \leq a$ except for a finite number of poles. Assume also that $R(\lambda)$ is regular in the two angular regions

$$F_1 : 0 \leq \arg(\lambda - c) \leq \theta_1 , \qquad F_2 : 0 \leq \pi - \arg(\lambda + c) \leq \theta_2$$

for some constants $c \geq 0$, $\theta_1 > 0$, $\theta_2 > 0$ with $\theta_1 + \theta_2 \leq \pi$, and assume that $R(\lambda)$ satisfies there

$$(5.2) \qquad |R(\lambda)|_X = O(e^{\alpha|\sin\theta_i\lambda|}) \text{ as } |\lambda| \to \infty \text{ in } F_i , \quad i = 1,2 ;$$

here α is a nonnegative constant. Let $u(t)$ be a solution of $Lu = 0$ with $|u|_X \in L_1$ for $t > 0$, and let $u_j = e^{i\lambda_j t} p_j(t)$ be the residue of $e^{it\lambda}R(\lambda)u(0)$ at the poles λ_j, $j = 1,\dots,m$ lying in the interior of the strip, i.e. in $0 < \text{Im}\,\lambda < a$; assume $\text{Im}\,\lambda_j < a - \varepsilon < a$, $j = 1,\dots,m$. Then $u(t)$ may be extended as a complex analytic function of $t = \sigma + i\tau$ (with values in X) into the angular region

$$(5.3) \qquad -\theta_1 < \arg(t - \alpha) < \theta_2$$

and satisfies there

$$|u(t) - \sum_1^m u_k(t)|_X \leq \text{constant } |u(0)|_Y e^{c|\tau|} \left(\frac{e^{-(a-\varepsilon)\mu_1}}{\mu_1} + \frac{e^{-(a-\varepsilon)\mu_2}}{\mu_2} \right),$$

where

$$\mu_1 = \sigma - \alpha + \tau \cot \theta_1 , \qquad \mu_2 = \sigma - \alpha - \tau \cot \theta_2 ,$$

and the constant depends only on the operator A.

Proof: We merely give a sketch of the proof since it is very similar to the proof of Theorem 2.2.

As before we extend $u(t)$ as zero for $t < 0$ and take Fourier transforms

$$\hat{u}(\lambda) = \frac{1}{i\sqrt{2\pi}} R(\lambda)u(0) , \qquad \lambda \text{ in resolvent set} .$$

This formula yields an extension of $\hat{u}(\lambda)$ as a meromorphic function into the region U = the union of F_1, F_2, and the strip $0 \leq \operatorname{Im} \lambda < a$. Since $|\hat{u}(\lambda)|_X = o(1)$ is $|\lambda| \longrightarrow \infty$, λ real, we find, on applying the Phragmén-Lindelöf principle in each F_i that $|\hat{u}(\lambda)|_X = O(e^{\alpha \operatorname{Im} \lambda})$ in F_i, $i = 1,2$. Proceeding as in the proof of Theorem 2.2 we set, for $t > \alpha$,

$$v(t) = \frac{1}{\sqrt{2\pi}} \int_{\Gamma} e^{it\lambda} \hat{u}(\lambda) d\lambda$$

where now Γ consists of a broken line segment consisting of a segment Γ_3: $\operatorname{Im} \lambda = a - \varepsilon$, $-c_2 \leq \operatorname{Re} \lambda \leq c_1$ (with endpoints on the sides of the two angular regions F_1, F_2, so that $c_i = c + (a-\varepsilon)\tan \theta_i$), and of the two infinite lines Γ_1, Γ_2 running from the endpoints of Γ_3 to infinity along the sides of F_1 and F_2. The integral is absolutely convergent, for on the infinite line segments Γ_1, Γ_2, of Γ,

$$|e^{it\lambda} \hat{u}(\lambda)|_X \leq \text{constant } e^{(\alpha - t)\operatorname{Im} \lambda} |u(0)|_Y .$$

Set $v_i(t)$ = the contribution of the integral over the line Γ_i, so that $v = v_1 + v_2 + v_3$. Clearly $v_3(t)$ is an entire function of t. We see furthermore that $v_1(t)$ may be extended as an analytic function of complex $t = \sigma + i\tau$ in the half plane $0 < \arg(t-\alpha) + \theta_1 < \pi$, for on Γ_1,

$$|e^{it\lambda}\hat{u}(\lambda)|_X \leq \text{constant } |u(0)|_Y \, e^{c\tau} e^{-\operatorname{Im}\lambda(\sigma-\alpha+\tau\cot\theta_1)} ,$$

while on Γ_2,

$$|e^{it\lambda}\hat{u}(\lambda)|_X \leq \text{constant } |u(0)|_Y \, e^{-c\tau} \, e^{-\operatorname{Im}\lambda(\sigma-\alpha-\tau\cot\theta_2)} ,$$

so that v_2 is analytic in the half plane $0 < \theta_2 - \arg(t-\alpha) < \pi$. Since, on Γ_3,

$$|e^{it\lambda}\hat{u}(\lambda)|_X \leq \text{constant } |u(0)|_Y \, e^{-\tau \operatorname{Re}\lambda - \sigma(a-\varepsilon)}$$

it follows that $v(t)$ is analytic in the angle $-\theta_1 < \arg(t-\alpha) < \theta_2$, and satisfies there: for $\sigma-\alpha+\tau\cot\theta_1 = \mu_1$, $\sigma-\alpha-\tau\cot\theta_2 = \mu_2$,

$$|v(t)|_X \leq \text{constant } |u(0)|_Y \, e^{c|\tau|}\left(\frac{e^{-(a-\varepsilon)\mu_1}}{\mu_1} + \frac{e^{-(a-\varepsilon)\mu_2}}{\mu_2}\right) .$$

The remainder of the proof is like that of Theorem 2.2. Our estimate $|\hat{u}(\lambda)|_X = O(e^{\alpha \operatorname{Im}\lambda})$ enables us to deform the contour Γ and we find, as before, that

$$u(t) = v(t) + \sum_1^m u_k(t) .$$

The theorem then follows from the preceding inequalities.

Remarks: 1) If in Theorem 2.3 we drop the assumption that $R(\lambda)$ is meromorphic in the strip, and simply assume it to be regular and satisfy (5.2) in F_1, F_2 we may still assert that the solution $u(t)$ is analytic in the angular region (5.3). For in the proof we may replace Γ by the broken line segment Γ' consisting of Γ_3': $-c \leq \lambda \leq c$, and the two infinite lines: $\operatorname{Im} \lambda = (|\operatorname{Re} \lambda| - c) \tan \theta_i$. The inverse Fourier transform of $\hat{u}(\lambda)$ on these infinite lines is treated as before, while the contribution from the segment Γ_3' is, of course, an entire function of t (of exponential type). See Chapter IV.

2) Clearly one may prove analogues of Corollary 1 of Theorem 2.2 and Corollary 1 of Theorem 2.1', giving asymptotic expansions for a solution of $Lu = 0$ — in case $R(\lambda)$ is also meromorphic in the whole upper half plane.

3) If, as in the remarks after Theorems 2.1 and 2.2, we are given an operator P with $PR(\lambda)u(0)$ meromorphic in the strip, and regular in the region F_i where it satisfies $|PR(\lambda)u(0)|_X = O(\;^{\alpha|\sin \theta_i \lambda|})$, $i = 1,2$, then P is analytic in the angular region (5.3) and satisfies an estimate similar to the one for $|u|_X$ in the theorem.

It is interesting to observe that under the conditions of Theorem 2.3 it is possible to give a lower bound for the norm $|u(t)|_Y$ of a solution u of $Lu = 0$ for large t. This is based on a device used by Krein and Prozorovskaya [1].

Before stating the result we observe that if $a = 0$ in the theorem, so that $R(\lambda)$ is regular on the real axis except for at

most a finite number of poles then the theorem yields the inequality in the region (5.3): for $\mu_1, \mu_2 \geq$ some positive constant μ,

$$(5.4) \qquad |u(t)|_X \leq \text{constant } \frac{1}{\mu} |u(0)|_Y \, e^{c|\tau|} .$$

The part of (5.3) in which $\mu_1, \mu_2 \geq \mu$ consists of the region (5.3) translated to the right by a distance μ.

We now state the result giving a lower bound for $|u(t)|_Y$ in the form of a "convexity theorem".

Theorem 2.4: Assume that $R(\lambda)$ satisfies the hypotheses of Theorem 2.3 with $a = 0$, and assume that $u(t)$ is a solution of $Lu = 0$ with $|u|_X \in L_1$ on $t > 0$. Let μ be a positive constant and let t_o be fixed. Set $\delta = (t_o-\alpha-\mu)/t$. (i) If $\theta_1 + \theta_2 < \pi$ then, for any positive number $\phi < (\theta_1+\theta_2)/2 = \theta$, there exist positive constants C_1, C_2, b depending only on A, θ_1, θ_2, ϕ and μ, such that

$$|u(t_o)|_X \leq C_1 C_2^{t_o} |u(t)|_Y^{\gamma} |u(0)|_Y^{1-\gamma} , \text{ for } 0 \leq \delta \leq 1$$

where

$$\gamma = b\delta^{\frac{\pi}{2\phi}} .$$

(ii) If $\theta_1 + \theta_2 = \pi$ there are constants C_1, C_2 depending only on A, θ_2 and μ such that

$$|u(t_o)|_X \leq C_1 C_2^{t_o} |u(t)|_Y^{\delta} \, |u(0)|_Y^{1-\delta}, \text{ for } 0 \leq \delta \leq 1 .$$

Thus if we fix t_o we see that for any solution $u(t)$, with $|u(t)|_X \in L_1$ on $t > 0$, there is a constant β (depending on the solution) such that for $t \geq t_o$

$$|u(t)|_Y \geq e^{-\beta t^{\frac{\pi}{2\phi}}} |u(0)|_Y \qquad \text{in case} \quad \theta_1 + \theta_2 < \pi ,$$

$$|u(t)|_Y \geq e^{-\beta t} |u(0)|_Y \qquad \text{in case} \quad \theta_1 + \theta_2 = \pi .$$

Proof: The term "constant" will always denote some constant depending only on A, ϕ, θ_1, θ_2 and μ. According to (5.4) $u(t)$ is analytic in the angular region G_1: $-\theta_1 \leq \arg (t-\alpha-\mu) \leq \theta_2$, and satisfies there, for $t = \sigma + i\tau$,

$$|u(t)|_X \leq \text{constant } |u(0)|_Y e^{c|\tau|} .$$

For some $T > 0$ we may apply the theorem to $u(t-T)$ and conclude that in the angular region G_2: $-\theta_1 \leq \arg (t-\alpha-\mu-T) \leq \theta_2$, $G_2 \subset G_1$,

$$|u(t)|_X \leq \text{constant } |u(T)|_Y e^{c|\tau|} .$$

Suppose $\theta_1 + \theta_2 < \pi$, set $\theta = (\theta_1+\theta_2)/2$, $\psi = (\theta_2-\theta_1)/2$. In G_1 we have

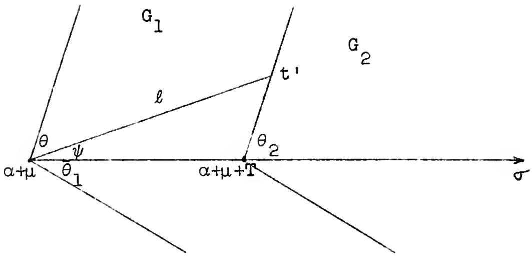
$$\mathrm{Re}\left(\frac{e^{-i\psi}}{\cos \theta} (t-\alpha-\mu)\right) \geq |t-\alpha-\mu| \geq |\tau|$$

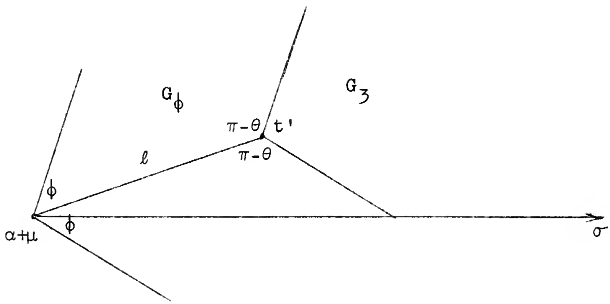
so that the function $w(t) = e^{-c(t-\alpha-\mu)e^{-i\psi}\sec \theta} u(t)$ satisfies

$$|w(t)|_X \leq \text{constant } |u(0)|_Y \quad \text{in } G_1 ,$$

$$|w(t)|_X \leq \text{constant } |u(T)|_Y \quad \text{in } G_2 .$$

Let us suppose now that $\theta_2 \geq \theta_1$; if $\theta_2 \leq \theta_1$, the argument is very similar. Through $\alpha+\mu$ draw a line segment ℓ (in the $t = \sigma + i\tau$ plane), with slope $\tan \psi$, until it touches the boundary of G_2 at some point t'.





By the law of sines ℓ has length $T\,\dfrac{\sin\theta_2}{\sin\theta}$. For $\phi < \theta$ let G_ϕ be the angular region $|\arg((t-\alpha-\mu)e^{-i\psi})| \leq \phi$, which has ℓ as its bisector at the vertex. Let G_3 be the angular region obtained by translating G_2 parallel to itself so that the new vertex is at t', i.e. G_3 is given by $-\theta_1 \leq \arg(t-t') \leq \theta_2$.

We now make a conformal transformation of variable by setting $z = z(t) = [(t-\alpha-\mu)e^{-i\psi}]^{\frac{\pi}{2\phi}}$. This maps G_ϕ onto the half plane

$\operatorname{Re} z \geq 0$. A (somewhat tedious) calculation shows that the line $\operatorname{Re} z = k = (T \sin \theta_2)^{\frac{\pi}{2\phi}} \left(\sin \frac{\theta - \phi}{1 - \frac{2\phi}{\pi}}\right)^{1 - \frac{\pi}{2\phi}} = d^{-1} T^{\frac{\pi}{2\phi}}$ lies in the image of the angle G_3 (d is here defined). Thus weconclude that the vector valued function $f(z) = w(t^{-1}(z))$ is analytic and bounded in the closed strip $0 \leq \operatorname{Re} z \leq k$ and satisfies

$$|f(z)|_X \leq \text{constant } |u(0)|_Y \quad \text{for} \quad \operatorname{Re} z = 0 \,,$$

$$|f(z)|_X \leq \text{constant } |u(T)|_Y \quad \text{for} \quad \operatorname{Re} z = k \,.$$

We now apply the "three lines theorem" and infer that for $0 \leq \operatorname{Re} z \leq k$

$$|f(z)|_X \leq \text{constant } |u(0)|_Y^{1 - \operatorname{Re} z/k} \, |u(T)|_Y^{\operatorname{Re} z/k} \,.$$

Returning to our t variable we see that for $t = t_o$ real, $\alpha + \mu \leq t_o \leq \alpha + \mu + T$, so that $\operatorname{Re} z/k = d \cos \frac{\pi\psi}{2\phi} \left(\frac{t - \alpha - \mu}{T}\right)^{\frac{\pi}{2\phi}}$, the inequality takes the form

$$|w(t_o)|_X \leq \text{constant } |u(0)|_Y^{1-\gamma} |u(T)|_Y^{\gamma} \,, \qquad \gamma = d \cos \frac{\pi\psi}{2\phi} \left(\frac{t_o - \alpha - \mu}{T}\right)^{\frac{\pi}{2\phi}} \,.$$

Thus

$$|u(t_o)|_X = e^{c(t_o - \alpha - \mu)\frac{\cos \psi}{\cos \theta}} \, |w(t_o)|_X$$

$$\leq \text{constant } e^{c(t_o - \alpha - \mu)\frac{\cos \psi}{\cos \theta}} \, |u(0)|_Y^{1-\gamma} |u(T)|_Y^{\gamma} \,.$$

Since T was arbitrary we have the desired result, with $C_2 = e^{c \frac{\cos \psi}{\cos \theta}}$ and $b = d \cos \frac{\pi\psi}{2\phi}$.

Consider now the case $\theta_1 + \theta_2 = \pi$. The region G_1 is now a half plane. The set of points in G_1 which are not in G_2 form a strip, with line ℓ (as above) perpendicular to its sides. If t_ℓ is the distance from any point t in the strip to the line ℓ we find that $|u(t)|_X e^{-ct_\ell \sin \theta}$ is bounded in the strip and that

$$|u(t)|_X \leq \text{constant } |u(0)|_Y e^{ct_\ell \sin \theta_2} \qquad \text{on the boundary of } G_1 ,$$

$$|u(t)|_X \leq \text{constant } |u(T)|_Y e^{c(t_\ell + T|\cos \theta_2|)\sin \theta_2} \qquad \text{on the boundary of } G_2 .$$

Applying again the "three lines theorem" in a slightly more refined form (see, for instance, I. I. Hirschman [1], Lemma 1) we obtain the following inequality for real t, $\alpha+\mu \leq t \leq \alpha+\mu+T$, so that $t_\ell = (t-\alpha-\mu)\sin \theta_2$,

$$|u(t)|_X \leq \text{constant } e^{ct_\ell \sin \theta_2} |u(0)|_Y^{1-\frac{t-\alpha-\mu}{T}} \left(e^{cT \sin \theta_2 |\cos \theta_2|} |u(T)|_Y\right)^{\frac{t-\alpha-\mu}{T}}$$

$$\leq \text{constant } C_2^t |u(0)|_Y^{1-\delta} |u(T)|_Y^{\delta} , \qquad \delta = \frac{t-\alpha-\mu}{T} \ ;$$

with $C_2 = e^{c(\sin^2\theta_2 + \sin \theta_2 |\cos \theta_2|)}$. Setting $t = t_o$, $T = t$ we obtain the desired inequality.

6. An abstract Weinstein principle

We now derive a theorem which may be called, following Lax, an abstract Weinstein principle (see P. D. Lax [3], §1.5 and A. Weinstein [1-2] where other references are also given).

Theorem 2.5: Let $u(t)$ be a solution of $Lu = 0$ on the whole t-axis, and assume that for a pair of real numbers a, b, $e^{at}|u(t)|_X$ belongs to L_1 on $t < 0$ and $e^{-bt}|u(t)|_X$ belongs to L_1 for $t > 0$. Then: (i) If $a+b < 0$, $u(t) = 0$. (ii) If $a+b = 0$ and if $R(\lambda)$ is regular on some interval on the line $\operatorname{Im} \lambda = a$ then $u(t) = 0$. (iii) If $a+b > 0$ assume that $R(\lambda)$ is meromorphic in the closed strip $-b \leq \operatorname{Im} \lambda \leq a$, and that for some constants k, α, $k > 0$, $0 < \alpha < \frac{\pi}{a+b}$, $|R(\lambda)| = O\left(e^{ke^{\alpha|\operatorname{Re} \lambda|}}\right)$ as $|\lambda| \to \infty$ in the strip. Then $u(t)$ is a finite sum of exponential solutions $u_j = e^{i\lambda_j t} p_j$, where λ_j are the poles of $R(\lambda)$ in the open strip $-b < \operatorname{Im} \lambda < a$. If, furthermore, each such pole λ_j has finite multiplicity then the set of solutions satisfying the above conditions is finite dimensional.

Note that $R(\lambda)$ is not assumed to be compact.

We recall (see Dunford-Schwartz [1], §VII.3) that if λ_o is a pole of $R(\lambda)$ of order r then λ_o has index r, i.e. the null spaces of $(\lambda_o - A)^j$ are strictly increasing with j until $j = r$ after which they remain constant. The dimension of the null space of $(\lambda_o - A)^r$ is called the multiplicity of λ_o.

Proof: By considering in place of u the function $w = e^{-bt}u(t)$, which satisfies $(L - ib)w = 0$ we may suppose that $b = 0$; note that $e^{(a+b)t}|w(t)|_X \in L_1$ for $t < 0$.

Supposing then that b = 0, set

$$\hat{u}_{\pm}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{\pm\infty} e^{-i\lambda t} u(t)dt .$$

$\hat{u}_+(\lambda)$ is analytic in the half plane Im $\lambda > 0$ and bounded and continuous in its closure, while $\hat{u}_-(\lambda)$ is analytic in the half plane Im $\lambda > a$ and continuous and bounded in its closure. Furthermore, in its relevant half plane we have $(\lambda - A)\hat{u}_{\pm} = \frac{1}{i\sqrt{2\pi}} u(0)$, or

$$\hat{u}_{\pm}(\lambda) = \frac{1}{i\sqrt{2\pi}} R(\lambda)u(0) , \quad \lambda \text{ in resolvent set.}$$

If $a \leq 0$ it follows that $\hat{u}_+(\lambda)$ and $\hat{u}_-(\lambda)$ are analytic extensions of each other, and hence define an entire bounded analytic function — which by Liouville's Theorem must be constant. Since, however, $|\hat{u}_+(\lambda)|_X \to 0$ as $\lambda \to \infty$ on the real axis it follows that this constant must be zero. Thus $\hat{u}_{\pm}(\lambda) \equiv 0$ and hence $u(t) = 0$.

Suppose then that $a > 0$. Since $R(\lambda)$ is meromorphic in the strip $0 \leq \text{Im } \lambda \leq a$ the formula above gives common analytic extensions of $\hat{u}_+(\lambda)$ and $\hat{u}_-(\lambda)$ into this strip as meromorphic functions, so that $\hat{u}_+$ and $\hat{u}_-$ are analytic extensions of each other. Let us denote by $w(\lambda)$ the meromorphic function in the whole plane defined by them. Since $\hat{u}_+$ ($\hat{u}_-$) is continuous on the line Im $\lambda = 0$ (a) it follows that the only possible poles of $w(\lambda)$ are the poles $\lambda_1, \ldots, \lambda_m$ of $R(\lambda)$ in the strip $0 < \text{Im } \lambda < a$.

By our hypothesis on $R(\lambda)$ we see that in the closed strip $0 \leq \text{Im } \lambda \leq a$, $|w(\lambda)|_X = O\left(e^{ke^{\alpha|\text{Re } \lambda|}}\right)$ as $|\lambda| \to \infty$, where

$0 < \alpha < \frac{\pi}{a}$, while on the boundary of the strip $|w(\lambda)|_X$ is bounded, in fact, $|w(\lambda)|_X = o(1)$ as $|\lambda| \longrightarrow \infty$ on the boundary of the strip (by the Riemann-Lebesgue lemma). Applying the Phragmén-Lindelöf theorem (see problems 333 and 339 in §3 of Pólya-Szegö [1]) we infer that $|w(\lambda)|_X = o(1)$ for $|\lambda| \to \infty$ in the strip.

Now let $R_j(\lambda)$ be the singular part of $R(\lambda)$ (in its Laurent expansion) at the pole λ_j. By the previous paragraph the function

$$w(\lambda) - \frac{1}{i\sqrt{2\pi}} \sum_j R_j(\lambda)u(0)$$

is a bounded entire function which tends to zero as $|\lambda| \longrightarrow \infty$ on the real axis. Hence this function is zero, so that

$$\hat{u}_{\pm}(\lambda) = \frac{1}{i\sqrt{2\pi}} \sum_j R_j(\lambda)u(0) \ .$$

Taking inverse Fourier transforms we find easily that $u(t)$ is equal to the sum of residues of $(e^{i\lambda t}R_j(\lambda)u(0))$ at λ_j, i.e. to a sum of exponential solutions. Q.E.D.

Corollary 1: Let $u(t)$ satisfy the conditions of Theorem 2.5 with L_1 replaced by L_p for some $p \geq 1$, or by L^0_∞. If $a + b < 0$ then $u(t) \equiv 0$. If $a + b \geq 0$ assume that $R(\lambda)$ is meromorphic in an open horizontal slit containing $-b \leq \operatorname{Im} \lambda \leq a$ and satisfies there the same growth condition at infinity as in the theorem. Then conclusions (ii) and (iii) of the theorem hold.

To prove the corollary we need only observe that for any positive ε, $e^{(a+\varepsilon)t}|u(t)|_X \in L_1$ on $t < 0$ and $e^{-(b+\varepsilon)t}|u(t)|_X \in L_1$ on $t > 0$. Using the theorem we conclude that u is a sum of

exponential solutions $e^{i\lambda_j^* t} p_j^*(t)$ where λ_j^* are poles of $R(\lambda)$ in the closed strip $-b \leq \operatorname{Im} \lambda \leq a$. Since, however, $e^{at}|u(t)|_X \in L_p$ or L_∞^o for $t < 0$ and $e^{-bt}|u(t)|_X$ belongs to L_p or L_∞^o for $t > 0$ we see that only the poles λ_j in the interior of the strip can make any contribution to this sum.

7. Completeness of exponential solutions

We turn now to the question of completeness of exponential solutions of $Lu = 0$ among all solutions satisfying some conditions on $t > 0$. We shall suppose $S = Y$ and set $R_S(\lambda) = R(\lambda)$. In §5, assuming that $R(\lambda)$ is meromorphic in $\operatorname{Im} \lambda \geq 0$, we have obtained an asymptotic expansion for $u(t)$ as a sum of exponential solutions. Thus the completeness of these solutions is tied up with the question whether there are solutions of $Lu = 0$ decaying faster than any exponential as $t \to \infty$. In Theorem 2.4 we have proved under rather strong conditions that no nontrivial solution can die down faster than $e^{-\beta t^p}$ for some β and p.

Our conditions on $R(\lambda)$ to ensure completeness, or to show that $u \equiv 0$ is the only solution decaying faster than every exponential, will involve the lower order ω of $R(\lambda)$ defined as follows:

Definition: $R(\lambda)$ as a map from Y into X (or Y) is of finite lower order $\omega \geq 0$ in the half plane $\operatorname{Im} \lambda \geq 0$ if for every $\varepsilon > 0$ there exists a sequence of differentiable Jordan arcs J_n lying in $\operatorname{Im} \lambda \geq 0$, with endpoints on the real axis on both sides of the origin, such that the distance of J_n from the origin tends to infinity and such that (i) $R(\lambda)$ exists on J_n and

$$|R(\lambda)|_{X\ (\text{or } Y)} \leq e^{|\lambda|^{\omega+\varepsilon}} \qquad \text{for} \quad \lambda \in J_n\ ;$$

(ii) ω is the smallest nonnegative number with this property.

We shall have need of the following result which we state for scalar valued analytic functions, but which may be extended easily to analytic functions with values in a Banach space.

Lemma 2.1: Assume that $f(\lambda)$ is analytic in the upper half plane $\operatorname{Im} \lambda > 0$ and continuous in its closure and that $|f(\lambda)| \leq M_1$ for λ real. Assume that on a sequence of half circular arcs J_n with origin as centers and endpoints on the real axis, and with radii $\to \infty$, $f(\lambda)$ satisfies

$$|f(\lambda)| \leq M_2 e^{\alpha|\lambda|} \qquad \text{on} \quad J_n\ ,$$

where α is a nonnegative constant. Then, if $M = \max\,(M_1, M_2)$,

$$|f(\lambda)| \leq M e^{\frac{4}{\pi}\alpha \operatorname{Im}\lambda} \qquad \text{for} \quad \operatorname{Im}\lambda \geq 0\ .$$

Proof: We may assume $M = 1$. We shall apply Theorem H of the Appendix in the book by Levinson [1] to one of the half circles J_n of radius r_n. According to this theorem we have for λ inside J_n (in case $|f| \leq 1$ on the real axis)

$$\frac{|\log f(\lambda)|}{\operatorname{Im}\lambda} \leq \frac{2r_n}{\pi} \int_0^{\pi} \log|f(r_n e^{i\phi})| \left[\frac{(r_n^2 - |\lambda|^2)\sin\phi}{|r_n^2 e^{2i\phi} - 2r_n e^{i\phi}\operatorname{Re}\lambda + |\lambda|^2|^2}\right] d\phi$$

$$\leq \frac{2}{\pi}\alpha r_n^2 \int_0^{\pi} [\qquad] d\phi$$

$$\to \frac{4}{\pi}\alpha \qquad\qquad \text{as } r_n \to \infty\,.$$

To illustrate the use of lower order we start with a simple result.

Theorem 2.6: *Let* $u(t)$ *be a solution of* $Lu = 0$ *with*

$$|u(t)|_Y = O(e^{-at}) \qquad \text{as } t \to \infty$$

for every real a. *Assume that* $R(\lambda)$ *satisfies either* (i) $R(\lambda)$ *is regular on a sequence of half circle arcs* J_n *in* $\operatorname{Im}\lambda \geq 0$ *with the origin as centers and radii tending to infinity, such that* $|R(\lambda)|_Y = O(e^{\frac{\pi\alpha}{4}|\pi|})$ *uniformly on the* J_n, *for some positive* α; *or* (ii) $R(\lambda)$, *as a map of* Y *into* Y, *is of lower order* $\omega \geq 1$ *in* $\operatorname{Im}\lambda \geq 0$, *and there exist non-overlapping differentiable arcs* $\gamma_1, \ldots, \gamma_k$ *issuing from a point on the real axis, and otherwise lying in* $\operatorname{Im}\lambda > 0$, *such that each of the* $k+1$ *regions into which* $\operatorname{Im}\lambda > 0$ *is divided by these arcs is contained in an angle with opening* $< \frac{\pi}{\omega}$. *Moreover* $R(\lambda)$ *exists on each* γ_j *for* $|\lambda|$ *sufficiently large and*

$$|R(\lambda)|_Y = O(e^{\alpha|\lambda|}) \qquad \text{for } |\lambda| \to \infty, \quad \lambda \in \gamma_j$$

for some constant $\alpha > 0$. *Then* $u(t) = 0$ *for* $t \geq \alpha$.

Note that $R(\lambda)$ is not assumed to be meromorphic.

Proof: Let $\hat{u}(\lambda)$ be the Fourier transform of $u(t)$ (extended as zero for $t < 0$). Then $\hat{u}(\lambda)$ is an entire function (in Y) of λ with bounded norm in the half plane $\operatorname{Im}\lambda \leq 0$; furthermore $|\hat{u}(\lambda)|_Y \to 0$ as $|\lambda| \to \infty$, λ real. Consider hypothesis (i); $\hat{u}(\lambda) = \frac{1}{i\sqrt{2\pi}} R(\lambda)u(0)$ then satisfies $|\hat{u}(\lambda)|_Y = O(e^{\frac{\pi\alpha}{4}|\lambda|})$ uniformly

on the J_n. By Lemma 2.1, $|\hat{u}(\lambda)|_Y \leq Ce^{\alpha \operatorname{Im} \lambda}|u(0)|_Y$ in $\operatorname{Im} \lambda \geq 0$, and it follows from the Paley-Wiener theorem that $u(t) = 0$ for $t \geq \alpha$.

In case (ii) we apply the Phragmén-Lindelöf principle to each of the regions in which $\operatorname{Im} \lambda > 0$ is divided by the arcs γ_j and deduce that $|\hat{u}(\lambda)|_Y = O(e^{\alpha|\lambda|})$ in the upper half plane. Applying a form of the Phragmén-Lindelöf theorem again (see the proof of Theorem 1.4.3 in Boas [1]) we see, in fact, that $|\hat{u}(\lambda)|_Y = O(e^{\alpha \operatorname{Im} \lambda})$ in $\operatorname{Im} \lambda \geq 0$, and the desired result follows once more with the aid of the Paley-Wiener theorem.

In §5 we obtained an asymptotic Fourier expansion

$$u(t) \sim \sum u_j(t)$$

for solutions of $Lu = 0$ in $t > 0$, where u_j is an exponential solution of index m_j corresponding to each λ_j (the poles of $R_S(\lambda)$ in $\operatorname{Im} \lambda > 0$). We shall now prove completeness of exponential solutions in the following sense: we give further conditions to ensure that under the same hypotheses giving asymptotic expansions, if we are given $\varepsilon > 0$ and $N > 0$, there exists a finite linear combination

$$\psi(t) = \sum_{j=1}^{n} \sum_{k=0}^{m_j-1} a_{jk} u_j^{(k)}(t) \tag{7.1}$$

(see the Introduction) such that

$$|u(t) - \psi(t)|_X \leq \varepsilon e^{-Nt} \qquad \text{for } t > \text{some fixed constant.}$$

The completeness results will be based on

Lemma 2.2: Let $u(t)$ be a solution of $Lu = 0$ with $|u(t)|_Y \in L_1$ on $t \geq 0$. Assume that $u(t)$ has an asymptotic Fourier expansion

$$u(t) \sim \sum u_j(t)$$

where $u_j = e^{i\lambda_j t} p_j$ is an exponential solution of index m_j, in the sense that there is some constant τ_0 such that, for any $N > 0$, there is a finite sum $\sum_1^k u_j$ with

$$\int_\tau^\infty |u(t) - \sum_1^k u_j|_Y dt \leq e^{-N\tau} \qquad \text{for } \tau \geq \tau_0 .$$

Assume that $R(\lambda)$ satisfies the conditions ((i) or (ii)) of Theorem 2.6. Then given $\varepsilon > 0$ there exists a finite linear combination $\psi(t)$ of the form (7.1) such that

$$|u(\alpha) - \psi(\alpha)|_Y \leq \varepsilon .$$

Proof: By the Hahn-Banach theorem this is equivalent to the statement that if h^* is any bounded linear functional on Y such that

(7.2) $\qquad h^*(u_k^{(j)}(\alpha)) = 0 \quad$ for $k = 0,\ldots,m_j-1$: $\;j = 1,2,\ldots$

then also $h^*(u(\alpha)) = 0$.

Let $\hat{u}(\lambda)$ be the Fourier transform of $u(t)$ (extended as zero for $t < 0$). Clearly $\hat{u}(\lambda)$ is a meromorphic function with poles at the λ_j. The function $h^*(\hat{u}(\lambda))$ is then a scalar valued analytic function in the compex λ plane with the exception, perhaps, of the

poles λ_j of $\hat{u}(\lambda)$ in Im $\lambda > 0$. However the coefficients of the negative powers of $(\lambda-\lambda_j)$ in the Laurent expansion of $\hat{u}$ about λ_j are vectors lying in the span $u_j(0), u_j^{(1)}(0), \ldots, u_j^{(m_j-1)}(0)$ (see the Introduction). Because of (7.2) $h^*(\hat{u}(\lambda))$ is then regular at λ_j, and consequently is an entire function of λ.

Next we observe that since $|\hat{u}(\lambda)|_Y$ is bounded in Im $\lambda \leq 0$ the same is true of $h^*(\hat{u}(\lambda))$. Moreover applying the Phragmén-Lindelöf principle as in the proof of Theorem 2.6 (recall that $\hat{u}(\lambda) = \frac{1}{i\sqrt{2\pi}} R(\lambda)u(0))$ we find $h^*(\hat{u}(\lambda)) = O(e^{\alpha \text{ Im } \lambda})$ in Im $\lambda \geq 0$. Since $h^*(\hat{u}(t)) = \widehat{h^*(u(\lambda))}$ it follows with the aid of the Paley-Wiener theorem that $h^*(u(t)) = 0$ for $t \geq \alpha$. In particular, $h^*(u(\alpha)) = 0$, and the lemma is proved.

Now for completeness

<u>Theorem 2.7</u>: <u>Let</u> $u(t)$ <u>be a solution of</u> $Lu = 0$ <u>with</u> $|u(t)|_X \in L_1$ <u>on</u> $t \geq 0$. <u>Assume that</u> $R(\lambda)$ <u>satisfies the conditions of Theorem</u> 2.2 <u>and is meromorphic in</u> Im $\lambda \geq 0$. <u>Assume furthermore that</u> $R(\lambda)$ <u>satisifes the hypotheses of Theorem</u> 2.6. <u>Let</u> $\sum_1^\infty u_j(t)$ <u>be the Fourier expansion for</u> $u(t)$ <u>where</u> $u_j(t) = e^{i\lambda_j t} p_j(t)$ <u>is an exponential solution of index</u> m_j <u>corresponding to the pole</u> λ_j <u>of</u> $R(\lambda)$. <u>Then, given</u> $\varepsilon > 0$ <u>and</u> $N > 0$ <u>there exists a finite linear combination</u> ψ <u>of the form</u> (7.1) <u>such that</u>

$$(7.3) \quad |D^j(u(t)-\psi(t))|_X \leq \varepsilon e^{-Nt} \quad \text{for} \quad t \geq \tau_j+\alpha \; ; \; j = 0,1,2,\ldots .$$

(τ_j <u>are constants in Theorem</u> 2.2, <u>and</u> α <u>is a constant in Theorem</u> 2.6.)

Proof: Suppose $\text{Im}\,\lambda_j > N$ for $j > m$ and consider $v(t) = u(t) - \sum_1^m u_j(t)$. Clearly $v(t)$ has as asymptotic expansion $\sum_{m+1}^{\infty} u_j(t)$. Applying the preceding lemma to $v(t)$ we find that there is a finite expansion of the form

$$\psi_1 = \sum_{m+1}^{n} \sum_{k=0}^{m_j-1} a_{jk} u_j^{(k)}(t)$$

such that

$$|v(\alpha) - \psi_1(\alpha)|_Y \le \frac{\varepsilon}{K_o} e^{-N\alpha} .$$

It follows from Theorem 2.2 applied for $t \ge \alpha$ that for $j = 0,1,\ldots$

$$|D^j(v(t) - \psi_1(t))|_X \le K_o e^{-N(t-\alpha)} |v(\alpha) - \psi_1(\alpha)|_Y \le \varepsilon e^{-Nt}, \quad t \ge \tau_j + \alpha ,$$

or

$$|D^j(u(t) - \sum_1^m u_j(t) - \psi_1(t)|_X \le \varepsilon e^{-Nt} , \qquad t \ge \tau_j + \alpha . \qquad \text{Q.E.D.}$$

Remarks: 1) It follows from Corollary 1 of Theorem 2.2 that if we require $u(t) \in L_p$ or L_∞^o instead of L_1, and if we assume that $R(\lambda)$ also satisfies the conditions of the corollary, then the inequality (7.3) still holds.

2) It is clear that if we assume $R(\lambda)$ to satisfy also the conditions of Theorem 1.5 then we can obtain such a completeness theorem for u in an angle in the complex t-plane.

It is of interest to investigate the completeness of (all) exponential solutions among solutions of $Lu = 0$ on a finite interval $|t| \le T$. We present a typical result in this direction. Up to now our conditions on the resolvent $R(\lambda)$ have been

(essentially) only in the upper half plane. We shall now impose conditions in both half planes (for convenience we shall make these symmetric in form about the real axis).

Consider a solution $u(t)$ on the interval $|t| \leq T$. On extending it as zero outside the interval, and taking Fourier transforms we find

$$(7.4) \qquad \hat{u}(\lambda) = \frac{1}{i\sqrt{2\pi}} R(\lambda)(e^{i\lambda T}u(-T) - e^{-i\lambda T}u(T)) \;, \qquad \lambda \text{ in resolvent.}$$

The function $\hat{u}(\lambda)$ is entire, of exponential type.

We shall prove an analogue of Theorem 2.7.

Theorem 2.8: Assume that $R(\lambda)$ is meromorphic in the entire plane, and regular in the region

$$(7.5) \qquad |\mathrm{Re}\ \lambda| \geq c \;, \qquad |\mathrm{Im}\ \lambda| \leq C \log |\mathrm{Re}\ \lambda| \;,$$

with $|R(\lambda)|_X = O(e^{\sigma|\lambda|})$ as $|\lambda| \to \infty$ in the region. Assume also that $|R(\lambda)|_X = O(e^{\beta|\mathrm{Im}\ \lambda|})$ as $\lambda \to \infty$ along the boundary of the region. Here c, C, σ, β are nonnegative constants. Let τ_o be a constant, $\tau_o > \beta + \frac{1}{C}$. Assume furthermore that $R(\lambda)$ and $R'(\lambda) = R(-\lambda)$ satisfy one of the conditions (i) or (ii) of Theorem 2.6. With α as given in Theorem 2.6 suppose that $\alpha + \tau_o + \frac{j_o}{C} < T$ for some integer $j_o \geq 1$ then, given any $\varepsilon > 0$, there is a finite sum $\psi = \sum_1^n u_k$ of exponential solutions such that

$$|D^j(u-\psi)|_X \leq \varepsilon \quad \text{on} \quad |t| \leq T - \alpha - \tau_o - j_o/C \qquad \text{for} \quad 0 \leq j \leq j_o \,.$$

Proof: We shall write u(t) as a sum of two solutions of Lu = 0 on overlapping intervals (τ_0-T,∞) and $(-\infty,T-\tau_0)$; each of these will be approximated by exponential solutions. We may suppose that $R(\lambda)$ has no poles on the real axis, for if it does we may always consider $v = e^{\delta t}u$ in place of u, for some small positive δ; v is then a solution of $\frac{1}{i}\frac{dv}{dt} - (A-i\delta)v = 0$, and by appropriate choice of δ the operator $(A-i\delta)$ will have no real points in its spectrum. So we suppose $R(\lambda)$ regular on the real axis. Taking inverse Fourier transforms in (7.4) we have

$$u(t) = \frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda(t+T)}R(\lambda)u(-T)d\lambda - \frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda(t-T)}R(\lambda)u(T)d\lambda \ .$$

Let Γ^+ (Γ^-) be the curve consisting of the segment $-c \leq \lambda \leq c$ and the two curves lying on the boundary of the region (7.5) in the upper (lower) half plane. We imagine the curves to have the orientation of increasing Re λ. As a first step in the proof we show that $u(t) = u_+(t) + u_-(t)$ for $|t| \leq T-t_0$, where

$$u_+(t) = \frac{1}{i\sqrt{2\pi}} \int_{\Gamma^+} e^{i\lambda(t+T)}R(\lambda)u(-T)d\lambda \ ,$$

$$u_-(t) = \frac{1}{i\sqrt{2\pi}} \int_{\Gamma^-} e^{i\lambda(t-T)}R(\lambda)u(T)d\lambda \ .$$

As in the proof of Theorem 2.2 one checks that these integrals are absolutely convergent. To obtain this formula we would like to deform the real axis into the contour Γ^+ or Γ^- as in the proof

of Theorem 1.4. However we no longer know that $|R(\lambda)u(-T)|_Y$ is bounded on the real axis, so that infinity is troublesome.

To get around this difficulty we introduce a

<u>Multiplier</u>: For some fixed number $r > 1$ define q for all values of λ in the complex plane, except for those $\lambda \neq 0$ which are purely imaginary, as follows

$$\begin{aligned} q(\lambda) &= e^{-\lambda^r} && \text{for } \operatorname{Re}\lambda > 0 \\ q(\lambda) &= q(-\lambda) && \text{for } \operatorname{Re}\lambda < 0 \\ q(0) &= 1 \end{aligned} \tag{7.6}$$

where, for $\operatorname{Re}\lambda > 0$ the principal value of λ^r is taken. Clearly $q(\lambda)$ is analytic in each half plane $\operatorname{Re}\lambda > 0$, $\operatorname{Re}\lambda < 0$ and, in any double angle $|\arg(\pm\lambda)| \leq \alpha$,

$$|q(\lambda)| \leq e^{-\cos(r\alpha)|\lambda|^r} . \tag{7.6$'$}$$

On the real axis, q is continuous and continuously once differentiable even at the origin. Furthermore $\frac{d^2q}{d\lambda^2}$ is absolutely integrable on the real axis. The function

$$j(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} q(\lambda)\, d\lambda , \qquad t \text{ real} ,$$

is therefore seen to be a C^∞ function, with $j(t) = O(t^{-2})$ as $|t| \to \infty$.

For any $\varepsilon > 0$ set $j_\varepsilon(t) = \varepsilon^{-1} j(t/\varepsilon)$; then $\hat{j}_\varepsilon = q(\varepsilon\lambda)$. It is easily verified that $j_\varepsilon(t)$ acts as a "mollifier", i.e. the

convolution of $j_\varepsilon(t)$ with any L_p function $u(t)$ tends to $u(t)$ in L_p as $\varepsilon \to 0$.

Consider now the convolution

$$j_\varepsilon * u = u_\varepsilon(t) \; . \tag{7.6)"}$$

Its Fourier transform is equal to $\hat{u}_\varepsilon(\lambda) = q(\varepsilon\lambda)\hat{u}(\lambda)$. Therefore we may write $u_\varepsilon(t) = u_{\varepsilon+}(t) + u_{\varepsilon-}(t)$ where

$$u_{\varepsilon+}(t) = \frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda(t+T)} q(\varepsilon\lambda)R(\lambda)u(-T)d\lambda \quad ,$$

$$u_{\varepsilon-}(t) = -\frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda(t-T)} q(\varepsilon\lambda)R(\lambda)u(T)d\lambda \; .$$

But now we are in a position to deform the contours, since $q(\varepsilon\lambda)$ dies down faster than any $e^{k|\lambda|}$ as $|\lambda| \to \infty$ in the region (7.5). Thus, in particular, for $|t| \le T-\tau_o$,

$$u_\varepsilon(t) = \frac{1}{i\sqrt{2\pi}} \int_{\Gamma^+} e^{i\lambda(t+T)} q(\varepsilon\lambda)R(\lambda)u(-T)d\lambda$$

$$- \frac{1}{i\sqrt{2\pi}} \int_{\Gamma^-} e^{i\lambda(t-T)} q(\varepsilon\lambda)R(\lambda)u(T)d\lambda \; .$$

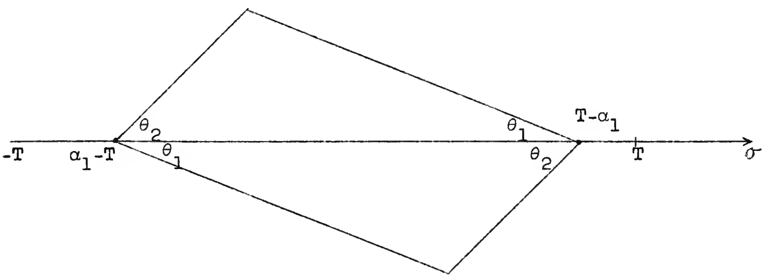
Having deformed the contours we may let $\varepsilon \to 0$ (since the resulting integrals are absolutely convergent) to obtain the desired decomposition.

From the proof of Theorem 2.2 we see that the functions $u_\pm(t)$ are continuously differentiable for $|t| \le T-\tau_o - \frac{1}{C}$; one verifies readily that they are solutions of $Lu = 0$. Thus the completeness

theorem would now follow by applying Theorem 2.7 to each solution $u_\pm(t)$ in turn on its corresponding semi-infinite interval — provided these solutions are absolutely integrable on these intervals. To verify this for, say, $u_+(t)$ we see that the contour Γ^+ may be deformed a bit more, so that its horizontal portion on the real axis is raised slightly above the axis. Since the new curve Γ^+ lies entirely in $\operatorname{Im} \pm \geq \delta$ for some $\delta > 0$ it follows easily that $|u_+(t)|_X = O(e^{-\delta t})$ as $t \to \infty$.

This completes the proof of the theorem.

Remark 1: Is we assume that $R(\lambda)$ and $R'(\lambda) = R(-\lambda)$ satisfy also the conditions of Theorem 2.3 then we can obtain such a completeness theorem for $u(t)$ in a region in the complex $t = \sigma + i\tau$ plane; namely the parallelogram (here α_1 is some constant).



It may occur that there are no nontrivial solutions of $Lu = 0$ on some interval. For example, let X consist of the C^1 functions on the interval $0 \leq x \leq 1$ vanishing at the origin; let Y be the space $C([0,1])$ and let $A = a\,\frac{d}{dx}$ where a is some constant $\neq 0$ which is not pure imaginary. Then the only solution $u(x,t)$ of

$$(7.7) \qquad Lu = (\frac{1}{i}\frac{\partial}{\partial t} - a\frac{\partial}{\partial x})u = 0$$

in $0 \le x \le 1$ and $|t| \le T$, with $u(0,t) = 0$ (i.e. with $u(x,t) \in X$ for fixed t) is $u \equiv 0$.

This example is related to the following

<u>Remark</u>: Suppose that $X = Y$, that A^{-1} exists as a bounded mapping of Y into itself, and the spectrum of A^{-1} consists only of the origin. Then the resolvent $R(\lambda)$ of A is regular in the whole plane, for $R(0) = A^{-1}$, while $R(\lambda) = (A^{-1} - \frac{1}{\lambda})^{-1}\frac{A^{-1}}{\lambda}$ for $\lambda \neq 0$. Thus there are no nontrivial exponential solutions of $Lu = 0$. It follows from Theorem 2.8 that if for arbitrary positive numbers β, C, α there exist constants $c = c(\beta, C)$, $\sigma = \sigma(\beta, C)$ such that the conditions of Theorem 2.8 are satisfied with these constants then the only solution $u(t)$ of $Lu = 0$ on any interval is $u \equiv 0$.

It is of interest to consider the preceding example (7.7) in case a is pure imaginary, $a = \frac{i}{\alpha}$ with, say, $\alpha > 0$. Then any solution $u(x,t)$ of $Lu = 0$ belonging to X on the interval $t_1 \le t \le t_2$ with $t_2 - t_1 > \alpha$ vanishes for $t \ge t_1 + \alpha$. A simple calculation shows that

$$(7.8) \qquad |R(\lambda)|_Y = O(e^{\alpha|\mathrm{Im}\,\lambda|}), \qquad |R(\lambda)|_X = O(\lambda e^{\alpha|\mathrm{Im}\,\lambda|}),$$

and Theorem 2.8 would then assure the (weaker) result that if $u \in X$ is a solution of (7.7) on $|t| \le T$ with $T > 2\alpha$ then u vanishes on the interval $|t| \le T - 2\alpha$.

This example, with $a = \frac{i}{\alpha}$, $\alpha > 0$ shows also that the conclusion $u(t) = 0$ for $t \ge \alpha$ cannot be improved, i.e. the solution need not vanish for $t < \alpha$.

Chapter III

Unique Continuation and Lower Bounds at Infinity

8. Finite Cauchy problem

In recent years the problem of uniqueness in the Cauchy problem for partial differential equations has received a great deal of attention. In particular, A. P. Calderón [1] has proved a very general uniqueness theorem. This has been simplified and extended by B. Malgrange [1] and L. Hörmander ([2], see also a forthcoming book by Hörmander). Some authors have treated the problem by setting it within the framework of a differential equation of the form (1) of (3) in some Banach space. (See, indeed, Lemma 1 in Calderón [1].) They prove uniqueness either for solutions vanishing at some value of t or vanishing with sufficient rapidity as $t \to \infty$. We mention, as an illustration, the following theorem due to P. D. Lax [1]; here $X = Y$ is a Hilbert space.

Theorem: *Let* $u(t)$ *be a solution of*

$$|Lu(t)|_X \leq \phi(t)|u(t)|_X \tag{8.1}$$

in $t \geq 0$ *such that the norms of* u *and* $\frac{du}{dt}$ *are square integrable*. Assume that there is a sequence of lines in the complex λ plane parallel to the real axis, $\operatorname{Im} \lambda = a_n$, with $a_n \to \infty$, on which the resolvent of A is uniformly bounded by a constant M. If $\phi(t) \leq c$ for some constant $c < M^{-1}$ then $|u(t)| = O(e^{at})$ for every real a implies that $u \equiv 0$.

The condition $c < M^{-1}$ cannot be dropped, for Lax presents an example with iD a self-adjoint operator, $M = \frac{1}{2}$ and a nonzero

solution $u(t)$ of $|Lu| \leq (\frac{1}{2} + \varepsilon)|u|$ (for any given $\varepsilon > 0$), such that $|u(t)|$ tends to zero like e^{-at^2} for some constant $a > 0$.

In this chapter we present a number of uniqueness results for the finite Cauchy problem (i.e. for solutions vanishing at some value of t for (1) and (3), and for solutions decaying rapidly at infinity; we also obtain, in some cases, explicit lower bounds for solutions.

The uniqueness of solutions of $Lu = 0$ dying down faster than any exponential at infinity is, as we have remarked in §7, closely connected with the completeness of exponential solutions and, we have already presented one result in that section, Theorem 2.6, concerning uniqueness for such solutions. Furthermore, in Theorem 2.4, we have derived lower bounds for nontrivial solutions of $Lu = 0$ under suitable conditions. These were obtained via convexity-like estimates which, in case (ii) of the theorem, assert, roughly, that $\log |u(t)|_X$ and $\log |u(t)|_Y$ are convex functions of t. Throughout this chapter, in deriving lower bounds for solutions, we attempt to do so via some kind of convexity statement. We believe that this is the natural framework for deriving these bounds unless the operator L is such that the backward Cauchy problem is well posed. (This is the problem of finding a solution $u(t)$ for $t \leq T$ with given values at $t = T$.) If it is well posed then there is an estimate $|u(0)|_X \leq$ constant $|u(T)|_X$ giving a lower bound for the norm of $u(T)$ in terms of the norm of $u(0)$. The operators that we treat are not, in general, those for which the backward Cauchy problem is well posed.

A. Beurling [1] (see Theorem 7 there and its application) has proved a general theorem concerning convolution operators that has application to partial differential equations, giving also lower bounds for the solutions as some variable goes to infinity. The theorem is based on an extension of the Wiener tauberian theorem.

For the sake of uniformity in treating the finite Cauchy problem, i.e. with $u(T) = 0$ for some T we shall prove backward uniqueness, i.e. for $t < T$.

We start with a simple result for the finite Cauchy problem. This is essentially the same as a result of Lyubič [1]. It holds also for weak solutions, as do a number of other results in this chapter, but we will consider only regular solutions. (It also admits an obvious extension to higher order differential equations.) We assume here that $X \subset Y$ are Banach spaces.

Theorem 3.1: _Let_ $u(t) \in X$ _be a solution of_ (1)

$$Lu = \left(\frac{1}{i}\frac{d}{dt} - A\right)u = 0$$

for $0 \leq t \leq T$, _with_ $u(T) = 0$. _Assume that there is a simple Jordan arc_ Γ _going to_ ∞ _lying in a closed angle in the open upper half_ λ-_plane on which_ $R(\lambda)$ _is defined and satisfies_

$$|R(\lambda)|_Y = O(e^{\alpha \operatorname{Im} \lambda})$$

for some constant $\alpha \geq 0$. _If_ $T > \alpha$ _then_ $u(t) = 0$ _for_ $t \geq \alpha$.

That we cannot say anything for $t < \alpha$ is seen by the last example in §7 — equation (7.7) with $a = \frac{i}{\alpha}$, $\alpha > 0$.

Proof: Extending $u(t)$ as zero outside the interval $0 \le t \le T$ and taking Fourier transforms, as we have done repeatedly, we find that the transform $\hat{u}(\lambda)$, which is an entire function of exponential type, satisfies

$$(\lambda - A)\hat{u}(\lambda) = \frac{1}{i\sqrt{2\pi}}\, u(0) \ ,$$

there is no contribution at $t = T$ since $u(T) = 0$. From our hypothesis it follows that $|u(\lambda)|_Y = O(e^{\alpha \operatorname{Im} \lambda})$ on Γ while on the real axis $|\hat{u}(\lambda)|_Y$ is bounded. Applying the Phragmén-Lindelöf theorem we conclude that $|u(\lambda)|_Y = O(e^{\alpha \operatorname{Im} \lambda})$ in the entire upper half plane. But then the Paley-Wiener theorem implies that $u(t)$ vanishes for $t \ge \alpha$.

Lyubič [1] also presents an interesting example showing that in some sense the result is best possible: If $\rho(\lambda)$ is a positive continuous function defined on $\lambda > 0$ such that $\lambda^{-1} \log \rho(\lambda) \longrightarrow \infty$ as $\lambda \longrightarrow \infty$ he constructs a Hilbert space Y, and in it a linear operator A, such that on some vertical half line $\operatorname{Re} \lambda = \lambda_o$, $\operatorname{Im} \lambda \ge$ constant, $R(\lambda)$ satisfies $|R(\lambda)|_Y \le \rho\,(\operatorname{Im} \lambda)$, and for which there is no uniqueness for the backward Cauchy problem for $Lu = 0$.

It is of interest to extend the theorem of Lax quoted above to a situation where the lines $\operatorname{Im} \lambda = a_n$ are allowed to contain some points of the spectrum — as might be the case, for instance, if iA were self adjoint. We shall present such a result for the finite Cauchy problem.

In the following whenever X = Y we shall not bother to write a subscript X or Y after the norm.

The following definitions and lemma will be used in this result and again later.

<u>Definition</u>: Let F be a family of horizontal infinite lines in the upper half of the complex λ plane. Let j be a nonnegative integer, and s, M be positive numbers. We shall say that the resolvent $R(\lambda) = (\lambda - A)^{-1}$ is (j,s)-bounded on F by M, if on each line of F the norm of $R(\lambda)$ is bounded by M for all λ on the line outside j intervals of length s, (their location, may differ from line to line of F).

We shall denote by E_2 the class of entire analytic functions $w(\lambda)$ (with values in X) which belong to L_2 on each horizontal line $\operatorname{Im} \lambda = a$ such that on each such line the L_2 norm of $|w(\lambda)|$ is bounded by $e^{|a|}$ times its L_2 norm on the real axis.

<u>Lemma 3.1</u>: <u>Let</u> Γ <u>consist of the real axis in the</u> λ-<u>plane minus</u> j (<u>nonoverlapping</u>) <u>intervals of length</u> s. <u>There is a constant</u> k <u>depending only on</u> j <u>and</u> s <u>such that for each function in</u> E_2 <u>the following inequality holds</u>

$$\int_{-\infty}^{\infty} |w(\lambda)|^2 d\tau \leq k \int_{\Gamma} |w(\lambda)|^2 d\lambda .$$

<u>Note that</u> k <u>is independent of the positions of the</u> j <u>intervals</u>.

The lemma is well-known, we briefly indicate its proof. If the lemma were false there would be a sequence of functions $w_n(\lambda)$ with $\int_{-\infty}^{\infty} |w_n(\lambda)|^2 d\lambda = 1$, and a sequence of axes Γ_n (with j intervals $I_{n1},\ldots,I_{nj}$ removed) such that

$$\int_{\Gamma_n} |w_n(\lambda)|^2 d\lambda \to 0 \ . \tag{8.2}$$

By choosing an appropriate subsequence (which we denote again by w_n), and by appropriate horizontal displacements we may suppose that $I_{11} = I_{21} = \ldots = I_{n1} = \ldots = I$ and that

$$\int_I |w_n(\lambda)|^2 d\lambda \leq \frac{1}{2j} \ .$$

We may also suppose that for each i the centers of the intervals I_{ni} converge (possibly to nonfinite values). Because of our hypothesis concerning E_1 the functions $w_n(\lambda)$ form a normal family and we conclude that a subsequence converges to a function $w(\lambda)$ in E_1 with $\int_I |w(\lambda)| d\lambda \geq \frac{1}{2j}$. From (8.2) however it follows that the $w_n(\lambda)$ converge to zero on some interval on the real axis. Hence $w(\lambda) \equiv 0$ — contradiction.

It is clear that much more general forms of this lemma hold and these can be employed in the same way as Lemma 3.1.

<u>Theorem 3.2</u>: <u>Let</u> $X = Y$ <u>be a Hilbert space and let</u> $u(t) \in X$ <u>be a solution of</u> (8.1)

$$|Lu(t)|_X \leq \phi(t)|u(t)|_X$$

<u>on the interval</u> $0 \leq t \leq T$ <u>with</u> $u(T) = 0$. <u>Let</u> F <u>be a sequence of lines</u> $\text{Im}\ \lambda = a_n$ <u>in the</u> λ-<u>plane with</u> $a_n \to \infty$ <u>and assume that</u> $R(\lambda)$ <u>is</u> (j,s)-<u>bounded on</u> F <u>by</u> M, <u>for some appropriate</u> j, s, M. <u>There is a constant</u> c <u>depending only on</u> j, s, <u>and</u> M <u>such that if</u> $|\phi(t)| \leq c$ <u>then</u> $u \equiv 0$.

Proof: It suffices to show that if $0 < \alpha < T$, $\alpha \leq 1$, then $u(t) = 0$ for $t \geq T-\alpha$. Thus, in fact, we may suppose (after a translation) that $T \leq 1$. Fix positive $\alpha < T$ and let $\zeta(t)$ be a nonnegative C^∞ function of t which vanishes for $t \leq \frac{\alpha}{2}$ and is equal to one for $t \geq \alpha$.

We shall establish the estimate

$$(8.3) \qquad \int_{\alpha/2}^{T} |e^{a_n t} u(t)|^2 dt \leq \text{constant } e^{2a_n \alpha}, \qquad n = 1,2,\ldots$$

with the constant independent of n; it follows easily that $u(t) = 0$ for $t \geq \alpha$. To this end set $u_1(t) = \zeta(t)u(t)$ and extend $u_1(t)$ as zero outside the interval $(0,T)$. For any fixed n set $e^{a_n t} u_1(t) = v(t)$ and denote the Fourier transform of v by $\hat{v}(\lambda)$. Since v has support in $(0,1)$ its transform belongs to the class E_2. Note that

$$(8.4) \qquad \int_{\alpha/2}^{T} |e^{a_n t} u(t)|^2 dt \leq \int_{-\infty}^{\infty} |v(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{v}(\lambda)|^2 d\lambda ,$$

by Parseval's theorem.

Setting $(L + ia_n)v(t) = f$ we see that, on taking Fourier transforms,

$$(\lambda + ia_n - A)\hat{v} = \hat{f} .$$

From our hypotheses it follows that for all real λ, except on j intervals of length s,

$$|\hat{v}(\lambda)| \leq M|\hat{f}(\lambda)| .$$

Thus if Γ is the real axis minus these intervals we have

$$\int_{\Gamma} |\hat{v}(\lambda)|^2 d\lambda \leq M^2 \int_{\Gamma} |\hat{f}(\lambda)|^2 d\lambda \ .$$

Applying the previous lemma we find that

$$\int_{-\infty}^{\infty} |\hat{v}(\lambda)|^2 d\lambda \leq kM^2 \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda \ ,$$

or, by Parseval's theorem

$$\int_{-\infty}^{\infty} |v(t)|^2 dt \leq kM^2 \int_{-\infty}^{\infty} |f(t)|^2 dt \ .$$

Now

$$(8.5) \qquad f(t) = \begin{cases} e^{a_n t} Lu & \text{for} \quad t \geq \alpha \\ e^{a_n t} (\zeta Lu - i \frac{d\zeta}{dt} u) & \text{for} \quad t \leq \alpha \end{cases}$$

and hence

$$\int_{-\infty}^{\infty} |v(t)|^2 dt \leq kM^2 \text{ constant } e^{2a_n \alpha} + kM^2 \int_{\alpha}^{\infty} \left| e^{a_n t} Lu \right|^2 dt$$

$$\leq kM^2 \text{ constant } e^{2a_n \alpha} + kM^2 c^2 \int_{\alpha}^{\infty} \left| e^{a_n t} u \right|^2 dt$$

by our assumption (8.1), or

$$\int_{-\infty}^{\infty} |v(t)|^2 \leq \text{constant } e^{2a_n \alpha} + kc^2 M^2 \int_{\alpha}^{\infty} |v|^2 dt \ .$$

Thus the desired inequality (8.3) follows. via (8.4), provided $kc^2M^2 < 1$, completing the proof.

Remark 1: It is clear that the preceding argument may be adapted to higher order differential operators L where L is a polynomial $L(\frac{1}{i}\frac{d}{dt})$ in $\frac{1}{i}\frac{d}{dt}$ of the form

$$L = (\frac{1}{i}\frac{d}{dt})^m + \sum_{r=1}^{m} (\frac{1}{i}\frac{d}{dt})^{m-r} A_r$$

with the A_r closed operators in X with ranges in X.
Assume, namely, that u is a solution in [0,T], with u(T) = 0, of

$$|Lu| \leq c \sum_{1}^{N} |P_j u|$$

where the $P_j u$ are differential operators of similar form. Assume that, say, P_1 is the identity operator. As above assume that on each line of the family F, outside of j intervals of length s, the operator $L(\lambda)$ has a bounded inverse on all of X and satisfies

$$|P_j(\lambda)L(\lambda)^{-1}|_X \leq M , \qquad j = 1,\ldots,N .$$

Then, if c is sufficiently small, the solution u is identically zero.

We conclude this section with

Theorem 3.3: *Let* X = Y *be a Hilbert space. Consider a solution* u *of* (8.1) *in* [0,T] *with* u(T) = 0. *Suppose that on a sequence of lines* Im $\lambda = a_n$, $a_n \rightarrow \alpha$, *the inequality*

(8.6) $$|R(\lambda)|_Y \leq \frac{M}{|\text{Re}(\lambda-\lambda_n)|^{\sigma}} \quad \text{for} \quad |\text{Re}\ (\lambda-\lambda_n)| \geq \frac{s}{2}$$

for some point λ_n with $\text{Im}\,\lambda = a_n$, and for some fixed positive number $\sigma \leq 1$. (Thus on the exterior Γ of an interval of length s on each line we assume that $|R(\lambda)|$ is not only bounded but decays at infinity.) If $\phi(t)$ is less than some constant c_o or if ϕ belongs to L_p on the interval, with $p\sigma > 1$, $2 \leq p < \infty$, then $u \equiv 0$.

Proof: As in the proof of Theorem 3.2 it suffices to show that for some $\varepsilon > 0$, $u = 0$ for $t > T-\varepsilon$; we may in fact suppose, after a displacement of the origin, that $T < \varepsilon$, and attempts to show that $u(t) = 0$ for $t > \frac{T}{2}$. Let ζ be a C^{∞} function which vanishes for $t \leq \frac{T}{4}$ and equals 1 for $t \geq \frac{T}{2}$. Set

$$r = \frac{2p}{p+2}, \qquad q = \frac{p+2}{2}.$$

Since $p \geq 2$, we have $1 \leq r \leq 2$. For n fixed define $v = e^{a_n t}u(t)$ on the interval and equal to zero outside and set $(L + ia_n)v = f$. Taking Fourier transforms as in the preceding theorem we find that

$$(\lambda + ia_n - A)\hat{v}(\lambda) = \hat{f}(\lambda)$$

so that if Γ is the complement of the interval $|\lambda - \text{Re}\,\lambda_n| < \frac{s}{2}$ we have by (8.6) (for $\frac{1}{q'} + \frac{1}{q} = 1$)

$$\int_{\Gamma} |\hat{v}(\lambda)|_Y^r d\lambda \leq M^r \int_{\Gamma} \frac{|\hat{f}(\lambda)|_Y^r}{|\lambda - \mathrm{Re}\ \lambda_n|^{\sigma r}} d\lambda$$

$$\leq M^r \left[\int_{\Gamma} \frac{d\lambda}{|\lambda - \mathrm{Re}\ \lambda_n|^{\sigma r q}} \right]^{\frac{1}{q}} \left[\int_{-\infty}^{\infty} |\hat{f}(\lambda)|_Y^2 d\lambda \right]^{\frac{p}{p+2}}$$

$$= \text{constant} \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^{\frac{r}{2}} .$$

With the aid of the theorem of the mean one verifies easily the inequality

$$\int_{-\infty}^{\infty} |\hat{v}(\lambda)|_Y^r d\lambda \leq C^r \int_{\Gamma} |\hat{v}(\lambda)|_Y^r d\lambda + C^r s^{r+1} \max_{\lambda \text{ real}} \left|\frac{d\hat{v}}{d\lambda}(\lambda)\right|_Y^r$$

where C is a constant depending only on r. Since the support of v(t) is in the interval $(0,\varepsilon)$ we also have

$$\left|\frac{d\hat{v}}{d\lambda}\right|_Y \leq \text{constant}\ \varepsilon \int_{-\infty}^{\infty} |v(t)|_Y dt .$$

Combining these inequalities we infer that

$$\int_{-\infty}^{\infty} |\hat{v}(\lambda)|_Y^r d\lambda \leq \text{constant} \left(\int_{-\infty}^{\infty} |f(t)|_Y^2 dt \right)^{\frac{p}{p+2}}$$

$$+ \text{constant}\ \varepsilon^r \left(\int_{-\infty}^{\infty} |v(t)|_Y dt \right)^r$$

where the constants depend only on M, s and r.

Since $r \leq 2$ we may apply the Hausdorff-Young inequality and infer that (for $\frac{1}{r} + \frac{1}{r'} = 1$),

$$\left[\int_{-\infty}^{\infty} |v(t)|_Y^{r'} dt\right]^{\frac{1}{r'}} \leq \text{constant} \left(\int_{-\infty}^{\infty} |f(t)|_Y^2 dt\right)^{\frac{1}{2}}$$

$$+ \text{constant } \varepsilon \int_{-\infty}^{\infty} |v(t)|_Y dt \, .$$

Hence by (8.5) (with $\alpha = \frac{T}{2}$) we have

$$\left(\int_{T/2}^{T} |v(t)|_Y^{r'} dt\right)^{\frac{1}{r'}} \leq \text{constant } e^{a_n T/2} + \text{constant} \left(\int_{T/2}^{T} |\phi v|_Y^2 dt\right)^{\frac{1}{2}}$$

$$+ \text{constant } \varepsilon^{1+\frac{1}{r}} \left(\int_{T/2}^{T} |v(t)|_Y^{r'} dt\right)^{\frac{1}{r'}} .$$

Now

$$\left(\int_{T/2}^{T} |\phi v|_Y^2 dt\right)^{\frac{1}{2}} \leq \left(\int_{T/2}^{T} |v|_Y^{r'} dt\right)^{\frac{1}{r'}} \left(\int_{T/2}^{T} \phi^p dt\right)^{\frac{1}{p}} .$$

Thus if ε is sufficiently small we find that

$$\left(\int_{T/2}^{T} |v(t)|_Y^{r'} dt\right)^{\frac{1}{r'}} \leq \text{constant } e^{a_n T/2} + \text{coefficient} \left(\int_{T/2}^{T} |v|_Y^{r'} dt\right)^{\frac{1}{r'}}$$

with a coefficient less than $\frac{1}{2}$. Hence

$$\left(\int_{T/2}^{T} |e^{a_n t} u|_Y^{r'} dt\right)^{\frac{1}{r'}} \leq \text{constant } e^{a_n T/2}$$

for every n. But then it follows that $u = 0$ for $t \geq \frac{T}{2}$. Q.E.D.

We observe that if in place of (8.6) we assume that on each line $\operatorname{Im} \lambda = a_n$,

$$|R(\lambda)|_Y \leq \frac{M}{1 + |\operatorname{Re}(\lambda - \lambda_n)|^{\sigma}} \tag{8.6$'$}$$

with fixed positive $\sigma \leq 1$ then the proof of the theorem works if the assumption $u(T) = 0$ is replaced by the weaker assumption that u is a solution for all $t > 0$ satisfying $|u(t)|_Y = O(e^{at})$ for any real a. (This condition implies that $\hat{v}(\lambda)$ (in the proof) is an entire function of λ.) Thus we have

Theorem 8.3': Let $X = Y$ be a Hilbert space and suppose that u is a solution of (8.1) in $(0,\infty)$ with $|u(t)|_Y = O(e^{at})$ for every real a. Suppose that (8.6)' holds in a sequence of lines $\operatorname{Im} \lambda = a_n$, $a_n \longrightarrow \infty$. If $\phi(t)$ belongs to L_p on $(0,\infty)$ with $p\sigma > 1$, $2 \leq p < \infty$, then $u(t) \equiv 0$.

9. Unique continuation

In this section we consider again solutions of (8.1) in a Hilbert space $X = Y$ for $t > 0$ and assume that the resolvent $R(\lambda)$ exists and is bounded on a sequence of lines $\operatorname{Im} \lambda = a_n$ except for a finite number of points on each line, near which $|R(\lambda)|$ becomes singular at a prescribed speed. We shall always assume here that $u(t)$ is a solution of (8.1) with $|u(t)| = O(e^{at})$ for every real a. Since $X = Y$ we shall not bother writing subscripts on the norms.

Theorem 3.4: Suppose that on each line $\operatorname{Im} \lambda = a_n$, $R(\lambda)$ satisfies

$$|R(\lambda)| \leq \frac{M}{|\operatorname{Re} \lambda - b_n|}$$

for some real number b_n. There exists a number c such that if $\phi(t) \leq \frac{c}{1+t}$ the solution u is identically zero.

Theorem 3.5: *Let* m *and* k *be nonnegative integers. Suppose that on each line* Im $\lambda = a_n$ *there are* m *points with real parts* $\lambda_{n1},\ldots,\lambda_{nm}$, *and that off these points* $R(\lambda)$ *satisfies*

$$(9.1)\qquad |R(\lambda)| \le M \prod_{j=1}^{m} \frac{(1+|\lambda-\lambda_{nj}|^2)^{k/2}}{|\lambda-\lambda_{nj}|^k}$$

for λ *on the line. If* $\phi(t) \le ce^{-\ t}$ *for some positive constants* c, σ *then* $u(t) = 0$ *for* $t >$ *a number* T *depending on* M, c *and* σ.

Theorem 3.5 is a considerable generalization of Theorem 1' in the appendix of Lax [1].

Proof of Theorem 3.4: It suffices to prove that for any positive number α

$$\int_{\alpha}^{\infty} |e^{a_n t}|^2 dt \le \text{constant } e^{2a_n\alpha}$$

with the constant independent of n. This implies that $u = 0$ for $t > \alpha$. Let $\zeta(t)$ be a C^∞ function which is zero for $t < \frac{\alpha}{2}$ and 1 for $t > \alpha$ and set

$$v(t) = e^{(a_n+ib_n)t}\zeta(t)u\ , \qquad t > 0\ ,$$

and $v(t) = 0$ for $t \le 0$. Thus we wish to establish the estimate

$$(9.2)\qquad \int_{\alpha}^{\infty} |v|^2 dt \le \text{constant } e^{2a_n\alpha}\ .$$

Set

$$f = [L + i(a_n + ib_n)]v$$

and take Fourier transforms. According to our hypothesis it follows that

$$|\lambda\hat{v}(\lambda)| \leq M|\hat{f}(\lambda)| \ .$$

By Parseval's identity

$$\int_0^\infty |\tfrac{dv}{dt}|^2 dt \leq M^2 \int_0^\infty |f(t)|^2 dt \ .$$

A simple integration by parts yields the inequality

$$\int_0^\infty |\tfrac{v}{t}|^2 dt \leq 2 \int_0^\infty |\tfrac{dv}{dt}|^2 dt$$

and inserting this into the preceding we find

$$\int_0^\infty |\tfrac{v}{t}|^2 dt \leq 2M^2 \int_0^\infty |f(t)|^2 dt$$

$$\leq \text{constant } e^{2a_n\alpha} + 2M^2 \int_\alpha^\infty |e^{a_n t} Lu|^2 dt$$

$$\leq \text{constant } e^{2a_n\alpha} + 2M^2c^2 \int_\alpha^\infty |\tfrac{v}{t}|^2 dt$$

by hypotheses. If $2M^2c^2 \leq \frac{1}{2}$ then the desired inequality (9.2) follows.

<u>Proof of Theorem 3.5</u>: Set $\tau = \sigma/2$ and let T be such that

(9.3) $$\phi(t) \leq \tfrac{1}{2}\left(\tfrac{\tau}{1+\tau}\right)^{mk} \frac{e^{-\tau T}}{M} \qquad \text{for } \ t \geq T \ .$$

Let α be any fixed number $> T$ and let ζ be a C^∞ function, as in the previous proof, vanishing for $t \leq \frac{\alpha}{2}$ equal to 1 for $t > \alpha$.

For fixed n set $v = e^{a_n t}\zeta u$, and $(L + ia_n)v = f$, so that

$$f = \begin{cases} e^{a_n t} Lu & \text{for} \quad t > \alpha \\ e^{a_n t}(\zeta Lu - \frac{1}{i}\frac{d\zeta}{dt} u) & \text{for} \quad t \le \alpha . \end{cases}$$

Considering ζu as zero for $t < 0$ we find, on taking Fourier transforms that

$$(\lambda + ia_n - A)\hat{v}(\lambda) = \hat{f}(\lambda) .$$

By our hypothesis we have, for λ real, $|w(\lambda)| \le M|\hat{f}(\lambda)|$, where

$$w(\lambda) = \prod_{j=1}^{m} \left(\frac{\lambda-\lambda_{nj}}{\lambda-\lambda_{nj}-i}\right)^k \hat{v}(\lambda)$$

is analytic in $\operatorname{Im}\lambda \le 0$.

If we denote by $|w|_d$ the L_2 norm of $|w(\lambda)|_Y$ on the line $\operatorname{Im}\lambda = -d$, which is clearly finite, then it is easily seen to be a decreasing function of d. Hence, in particular,

$$|w|_\tau \le |w|_0 \le M|\hat{f}|_0 .$$

On $\operatorname{Im}\lambda = -\tau$ the absolute value of the factor $\frac{\lambda-\lambda_{nj}}{\lambda-\lambda_{nj}+i}$ is at least $\frac{\tau}{1+\tau}$, so that

$$|\hat{v}|_\tau \le (\tfrac{1+\tau}{\tau})^{mk} M|\hat{f}|_0 .$$

Setting $M^2(\frac{1+\tau}{\tau})^{2mk} = M_1$, the previous inequality is equivalent, via Parseval's identity, to the inequality

$$\int_0^\infty \left|e^{(a_n-\tau)t}\zeta u\right|^2 dt \le M^2\left(\frac{1+\tau}{\tau}\right)^{2mk} \int_0^\infty |f(t)|^2 dt$$

$$\le \text{constant } e^{2a_n\alpha} + M^2\left(\frac{1+\tau}{\tau}\right)^{2mk} \int_\alpha^\infty \left|e^{a_n t} Lu\right|^2 dt.$$

From (9.3) it follows that

$$\int_\alpha^\infty \left|e^{(a_n-\tau)t} u\right|^2 dt \le \text{constant } e^{2a_n\alpha} + \frac{1}{4}\int_\alpha^\infty \left|e^{(a_n-\tau)t} u\right|^2 dt ,$$

so that

$$\int_\alpha^\infty \left|e^{(a_n-\tau)t} u\right|^2 dt \le \text{constant } e^{2a_n\alpha} ,$$

with the constant independent of n. Consequently $u = 0$ for $t > \alpha$.

Q.E.D.

10. Convexity and lower bounds

We turn now to our program of obtaining lower bounds for solutions of (8.1) via convexity theorems. In this section, and the remainder of this chapter we shall assume that $X = Y$ is a Hilbert space, with $(\ ,\ )$ denoting scalar product; we shall omit writing subscripts X or Y on the norms. In Theorems 3.6 and 3.7 below we assume that the solutions u have strong first and second derivatives.

The first example of convexity is the following known elementary result.

Theorem 3.6: *Let* $u(t)$ *be a solution of*

(10.1) $$Lu = \left(\frac{1}{i}\frac{d}{dt} - A\right)u = 0$$

where $A = \gamma B$, γ *is a constant*, B *is a symmetric operator*. *Then* $\log |u(t)|$ *is a convex function of* t.

To see this one simply observes, on differentiating, that the second derivative of $\log (|u(t)|^2)$ is nonnegative.

This simple trick can be carried over to certain equations (10.1) with $A = A(t)$ depending on t; we shall carry out the details of the computation for one such case. The conditions we impose are such as arise in parabolic differential equations (where $A(t)$ is an elliptic operator acting on other variables).

Writing $iA(t) = B(t)$ we shall assume

(i) $B(t)$ is a closed densely defined operator for each t, and that $u(t)$ belongs to the domain of $B^*(t)$ as well as to that of $B(t)$.

(ii) We assume some smoothness of $B(t)$ in its dependence on t and assume also that B is "almost self adjoint". These hypotheses are best expressed in a single condition: if $u(t)$ is the solution then for some positive constants k, c

$$\operatorname{Re} \frac{d}{dt} (B(t)u(t),u(t)) \geq \frac{1}{2} |(B+B^*)u|^2 + c \operatorname{Re} ((B-k)u,u) .$$

(This condition is, for instance, automatically satisfied if $B(t)$ is a bounded self adjoint operator having bounded derivative, and such that $kI - B(t)$ is a positive operator.)

Theorem 3.7: *Let* $u(t)$ *be a solution of* $Lu = 0$, *i.e.*

$$\frac{du}{dt} - Bu = 0 , \qquad\qquad 0 \leq t \leq T ,$$

satisfying conditions (i)-(ii). Then the function $\log |e^{-kt}u(t)|$ is a convex function of the variable $s = e^{ct}$.

From the convexity it follows, by a direct computation that if $0 < t_1 < t$ then

$$e^{-kt}|u(t)| \geq |u(0)| \left[\frac{|u(t_1)|}{|u(0)|} e^{-kt_1} \right]^{(e^{ct}-1)/(e^{ct_1}-1)}$$

Thus we find, on fixing t_1, that

$$e^{-kt}|u(t)| \geq \rho |u(0)| \rho^{-a^t} .$$

Here the number ρ depends on the particular solution, and $a = e^c$. We should observe that if $(kI - B)u,u) \geq 0$ then $\rho \geq 1$, for $e^{-kt}|u(t)|$ is then decreasing, (see (10.2) below).

Our conditions (i), (ii), condition (ii) being a rather restrictive one, are closely related to the conditions assumed by J.L. Lions and B. Malgrange [1] in their proof of uniqueness for backward parabolic equations. They prove uniqueness for the finite Cauchy problem while we actually give lower bounds for the solution. However, their uniqueness proof is valid for weak solutions of the equation.

Protter [1], and Lees [1] have proved various unique continuation results at infinity for parabolic differential equations (other references may be found there); see also Protter [2] for asymptotic results on hyperbolic equations.

<u>Proof of Theorem 3.7</u>. Set $v(t) = e^{-kt}u(t)$. To show that $\log |v|^2$ is a convex function of s it suffices to show that, for $q = (v,v)$,

$$q \frac{d^2q}{ds^2} \geq \left(\frac{dq}{ds}\right)^2 , \quad \text{or} \quad \ddot{q} \geq c\dot{q} + \frac{1}{q}\dot{q}^2 ;$$

here $\frac{d}{dt}$ is denoted by a dot. The function v is a solution of the differential equation

$$\dot{v} = B(t)v - kv ,$$

and therefore

$$\dot{q} = 2\ \mathrm{Re}(\dot{v},v) = 2\ \mathrm{Re}(Bv,v) - 2k(v,v)$$

(10.2) $$= 2\ \mathrm{Re}\ e^{-2kt}(Bu,u) - 2kq ,$$

$$\ddot{q} = -4k\ \mathrm{Re}(Bv,v) + 2e^{-2kt} \frac{d}{dt} \mathrm{Re}(Bu,u) - 2k\dot{q} .$$

Therefore

$$\ddot{q} - \frac{1}{q}\dot{q}^2 - c\dot{q} = -2k\dot{q} - 4k\ \mathrm{Re}(Bv,v) + 2e^{-2kt} \frac{d}{dt} \mathrm{Re}(Bu,u) - \frac{((B+B^*)v,v)^2}{(v,v)}$$

$$+ 8k\ \mathrm{Re}(Bv,v) - 4k^2(v,v) - c\dot{q}$$

$$\geq 2e^{-2kt} \frac{d}{dt} \mathrm{Re}(Bu,u) - |(B+B^*)v|^2 - c\dot{q}$$

by Schwarz' inequality. By (ii) this last expression is non-negative. Q.E.D.

It is clear that the theorem may be generalized slightly. For instance, we may permit k and c to vary with t. We will then obtain a different lower bound for $|u(t)|$.

P. Cohen and M. Lees [1] have proved an interesting result giving lower bounds for solutions, in a Hilbert space, of (8.1)

(10.3) $$|Lu| = \left|\frac{du}{dt} - iAu\right| \leq \phi(t)|u| , \qquad t > 0 ,$$

assuming iA to be a symmetric operator (independent of t). They showed, namely, that if $\phi(t)$ belongs to L_p, for some p with $1 \leq p \leq 2$, then there are constants K, μ such that

$$|u(t)| \geq Ke^{-\mu t} .$$

We shall give a somewhat simpler derivation of this result (assuming, however, self adjointness) together with some extensions via a convexity type argument. P. Cohen has informed us that the method used in their paper yields also various extensions of the result.

Our main assumption is that $iA = B + iH$ where B and H are self adjoint operators such that B and the unitary operator e^{iH} commute. As was pointed out to us by I.E. Segal we may, in fact, assume that H = 0 for if u satisfies (10.3) then $v = e^{itH}u$ satisfies

$$(10.3)' \qquad \left|\frac{dv}{dt} - Bv\right| \leq \phi(t)|v| \quad \text{and} \quad |v| = |u| .$$

Thus we shall assume that iA = B is self adjoint.

We use the following

<u>Lemma 3.2.</u> <u>Let</u> v(t) <u>be defined on</u> $a \leq t \leq b$, <u>belong to the domain of</u> B <u>for every</u> t <u>and have a strong derivative</u>. <u>Then the following inequality holds</u>

$$(10.4) \qquad \max_{a\leq t\leq b} |v(t)|^2 \leq 2(|v(a)|^2 + |v(b)|^2) + 4\left(\int_a^b \left|\frac{dv}{dt} - Bv\right| dt\right)^2 .$$

<u>Proof</u>. Let E be the projection operator in X associated with the positive part of the spectrum of B, and set $v_1 = Ev$, $v_2 = (I-E)v$. If $\frac{dv}{dt} - Bv = f$ then $(\frac{d}{dt} - B)v_1 = Ef$; $(\frac{d}{dt} - B)v_2 = (I-E)f$, so that

$$\frac{d}{dt}(v_1,v_1) = 2\,\mathrm{Re}(Bv_1,v_1) + 2\,\mathrm{Re}(Ef,v_1) ,$$

with a similar relation holding for v_2. Hence we obtain the inequalities

$$\frac{d}{dt}(v_1,v_1) \geq 2\,\mathrm{Re}(Ef,v_1) , \qquad \frac{d}{dt}(v_2,v_2) \leq 2\,\mathrm{Re}((I-E)f,v_2) ,$$

so that

$$(v_1(b),v_1(b)) - (v_1(t),v_1(t)) \geq 2\,\mathrm{Re}\int_t^b (Ef(s),v_1(s))ds ,$$

or

$$|v_1(t)|^2 \leq |v_1(b)|^2 + 2V\int_t^b |f(t)|dt$$

and, similarly,

$$|v_2(t)|^2 \leq |v_2(a)|^2 + 2V\int_a^t |f(t)|\,dt ,$$

where $V = \max_{a\leq t\leq b} |v(t)|$. Adding, we find

$$|v(t)|^2 \leq |v_1(b)|^2 + |v_2(a)|^2 + 2V\int_a^b |f(t)|dt ,$$

or

$$V^2 \leq |v(b)|^2 + |v(a)|^2 + 2V\int_a^b |f(t)|dt$$

from which (10.4) follows.

Before stating the main result we apply the lemma to solutions of $(10.3)'$. Suppose that $u(t)$ is a solution of $(10.3)'$ on an interval $a \le t \le b$ on which $\int_a^b \phi(t)dt \le \frac{1}{2\sqrt{2}}$ then we claim that

$$(10.5) \qquad |u(t)| \le 2\sqrt{2}\, |u(a)|^{\frac{b-t}{b-a}}\, |u(b)|^{\frac{t-a}{b-a}}, \qquad a \le t \le b .$$

This is the convexity-like statement from which lower bounds for the solution will follow.

To prove (10.5) set $w(t) = e^{\sigma t}u(t)$ with σ real. Then

$$\left|\frac{dw}{dt} - (B+\sigma)w\right| \le e^{\sigma t}\left|\frac{du}{dt} - Bu\right| .$$

Applying the lemma with B replaced by $B+\sigma$ we obtain the inequality

$$\max |e^{\sigma t}u(t)|^2 \le 2|e^{\sigma a}u(a)|^2 + 2|e^{\sigma b}u(b)|^2 + 4\left(\int_a^b |e^{\sigma s}Lu(s)|ds\right)^2$$

$$\le 2|e^{\sigma a}u(a)|^2 + 2|e^{\sigma b}u(b)|^2 + \frac{1}{2}\max |e^{\sigma t}u(t)|^2$$

by $(10.3)'$ and our hypothesis on ϕ. Hence

$$|e^{\sigma t}u(t)|^2 \le 4|e^{\sigma a}u(a)|^2 + 4|e^{\sigma b}(b)|^2 .$$

Choosing σ so that the two terms on the right become equal gives the desired inequality (10.5).

Let us now assume that $\phi(t)$ is integrable on every finite interval. Starting with $t_o = 0$ let t_n, $n = 1,2,\ldots$ be such that

$$\int_{t_{n-1}}^{t_n} \phi(t)dt = \frac{1}{6\sqrt{2}} ,$$

and set $t_{n+1} - t_n = \rho_n$. If there are only a finite number of such intervals the last has infinite length, and the integral of ϕ over it does not exceed $\frac{1}{6\sqrt{2}}$.

Theorem 3.8: Let u be a solution of (10.3). (i) If $\phi(t) \in L_p$ for some p, $1 \leq p \leq 2$ then

$$|u(t)| \geq |u(0)| e^{-\mu t_\beta t} \qquad \text{for} \quad t \geq t_2 .$$

(ii) If $\phi(t) \in L_p$ for some p, $2 < p \leq \infty$ then

$$|u(t)| \geq |u(0)| e^{-\mu t^{(2-2/p)}\beta t}, \qquad t \geq t_2 .$$

(iii) If $\phi(t) \leq Kt^{\kappa}$ then

$$|u(t)| \geq |u(0)| e^{-\mu t^{2\kappa+3}\beta t}, \qquad t \geq t_2 .$$

These are all special cases of

(iv) Suppose that for some numbers k, K

$$\sum_0^n \frac{1}{\rho_j} \leq K\left(\sum_0^n \rho_j\right)^k \qquad n = 1,2,\ldots \tag{10.6}$$

then

$$|u(t)| \geq |u(0)| e^{-\mu t^{k+1}\beta t}, \qquad t \geq t_2 . \tag{10.7}$$

In each case μ is a fixed constant while β is a constant depending on the solution.

Proof: We shall first indicate how (i)-(iii) follow from (iv) and then prove (iv).

Observe that if $\phi \in L_p$ then

$$\frac{1}{6\sqrt{2}} = \int_{t_n}^{t_{n+1}} \phi \, dt \leq \left(\int_{t_n}^{t_{n+1}} \phi^p dt \right)^{\frac{1}{p}} \rho_n^{\frac{p-1}{p}} ,$$

from which it follows that

$$\sum \rho_n^{1-p} < \infty .$$

For $p \leq 2$ it follows that $\sum \rho_j^{-1} < \infty$, so that (10.6) holds with $k = 0$. Thus (i) follows from (iv).

For $p > 2$ we have, by Hőlder's inequality

$$\sum_0^n \frac{1}{\rho_j} = \sum_0^n \frac{\rho_j}{\rho_j^2} \leq \left(\sum_0^n \frac{1}{\rho_j^{p-1}} \right)^{\frac{2}{p}} \left(\sum_0^n \rho_j \right)^{1-\frac{2}{p}} ,$$

so that (10.6) holds with $k = 1 - \frac{2}{p}$; and (ii) follows from (iv).

In case (iii) we have

$$\frac{1}{6\sqrt{2}} = \int_{t_j}^{t_{j+1}} \phi \, dt \leq K t_{j+1}^{\kappa} \rho_j \leq K t_{n+1}^{\kappa} \rho_j \qquad \text{for} \quad j \leq n .$$

Thus

$$\frac{n+1}{6\sqrt{2}} \leq K t_{n+1}^{\kappa} \sum_0^n \rho_j = K t_{n+1}^{\kappa+1}$$

and

$$\sum_0^n \frac{1}{\rho_j} \leq (n+1) K \, 6\sqrt{2} \; t_{n+1}^{\kappa} \leq \text{constant } t_{n+1}^{2\kappa+2}$$

so that (10.6) holds with $k = 2\kappa+2$.

We have finally to prove (iv). We first prove (10.7) for $t = t_n$, $n = 2,3,\ldots$. To this end we apply (10.5) with $t = t_j$,

$a = t_{j-1}$ and $b = t_{j+1}$. Taking logarithms, and writing $\log |u(t_j)| = \sigma_j$, this inequality has the form

$$\sigma_j \leq \log 2\sqrt{2} + \frac{\rho_j}{\rho_j + \rho_{j-1}} \sigma_{j-1} + \frac{\rho_{j-1}}{\rho_j + \rho_{j-1}} \sigma_{j+1} ,$$

or, after multiplying by $(\frac{1}{\rho_j} + \frac{1}{\rho_{j-1}})$,

$$\frac{\sigma_j - \sigma_{j-1}}{\rho_{j-1}} \leq \log 2\sqrt{2} \left(\frac{1}{\rho_j} + \frac{1}{\rho_{j-1}}\right) + \frac{\sigma_{j+1} - \sigma_j}{\rho_j} .$$

Summing over j from 1 to n we find

$$\frac{\sigma_1 - \sigma_o}{\rho_o} \leq \log 2\sqrt{2} \sum_1^n \left(\frac{1}{\rho_j} + \frac{1}{\rho_{j-1}}\right) + \frac{\sigma_{n+1} - \sigma_n}{\rho_n} ,$$

or

$$\sigma_{n+1} - \sigma_n + 2 \log 2\sqrt{2}\, \rho_n \sum_0^n \frac{1}{\rho_j} \geq \rho_n \frac{\sigma_1 - \sigma_o}{\rho_o}$$

Summing again over n from 0 to N we find

$$\sigma_{N+1} - \sigma_o + 2 \log 2\sqrt{2} \left(\sum_0^N \frac{1}{\rho_j}\right)\left(\sum_0^N \rho_n\right) \geq \frac{\sigma_1 - \sigma_o}{\rho_o} \sum_0^N \rho_n ,$$

or, on taking exponentials, and using (10.6),

$$\frac{|u(t_{N+1})|}{|u(0)|} \geq 8^{-Kt_{N+1}^{k+1}} \left(\frac{|u(t_1)|}{|u(t_o)|}\right)^{t_{N+1}/t_1} , \qquad N \geq 1 .$$

This is the desired form (10.7) with $\mu = K \log 8$, $\beta = \left(\frac{|u(t_1)|}{|u(t_o)|}\right)^{1/t_1}$.

To establish the same estimate for any other value of $t \geq t_2$ suppose that $t_n < t < t_{n+1}$ for $n \geq 2$. Then in the preceding

argument we may delete the points t_n and t_{n+1} and replace them by the single point t. Since $t_{n+2} - t_{n-1} \leq \frac{3}{6\sqrt{2}} = \frac{1}{2\sqrt{2}}$ we may still carry out the same argument, and since $\sum \frac{1}{\rho_j}$ is not increased by this alteration the proof is unchanged and we thus obtain the desired estimate at t (which is now one of our selected points) with the same values of μ and β.

This completes the proof of Theorem 3.8.

In case (ii) with $p = \infty$ the inequality states that

$$|u(t)| \geq |u(0)|e^{-\mu t^2} \beta^t .$$

The example of P. Lax [1], cited on page 76-77, shows that the exponent 2 cannot be improved.

11. Another lower bound

In this section we present an attempt to find a lower bound for solutions of (8.1) in the framework of Lax's theorem at the beginning of §8. The result is rather special and its proof, which uses several ideas occurring in previous sections is somewhat complicated.

We shall consider a solution of (8.1) on $0 \leq t < T$, where T might be infinite, with $\phi(t) \leq$ constant c. Let F be the family of horizontal lines in the upper half λ-plane, Im $\lambda \geq 0$. Recalling the definition of §8 we shall assume that $R(\lambda)$ is (j,s) bounded on F by M. (Thus F may be the multiple of a self adjoint operator, as in the previous result.) We have not succeeded in obtaining a

lower bound for $|u(t)|$; however, given any positive number ρ, we shall show how to obtain a lower bound for

$$\int_t^{t+\rho} |u(s)|^2 ds .$$

Again the proof is based on a convexity-like argument.

<u>Theorem 3.9</u>: <u>There is a number</u> c_o <u>depending on</u> j, s, M <u>and</u> ρ <u>such that if</u> $c \leq c_o$ <u>then</u>

$$(11.1) \qquad \int_t^{t+\rho} |u(s)|^2 ds \geq K_o K_1^t e^{-\beta_o \mu^t} .$$

<u>Here</u> K, <u>and</u> $\mu \geq 1$, <u>are fixed constants depending on</u> j, s, M <u>and</u> ρ, <u>and</u> β_o <u>is a constant depending on the solution</u>.

<u>Proof</u>: (i) We may assume that $\rho \leq 1/2$. Set $\rho = 3d$,

$$t_n = 2nd , \qquad n = 0,1,2,\ldots,$$

and let I_n denote the interval (t_n, t_n+d). In order to prove (11.1) it suffices to prove the inequalities

$$(11.1)' \qquad \int_{t_n}^{t_n+d} |u(s)|^2 ds \geq K K_1^{t_n} e^{-\beta\mu^{t_n}}$$

with some constants K, K_1, β, μ; for any interval $(t,t+\rho)$ contains at least one of these intervals I_n and hence (11.1) follows - with $\beta_o = \beta\mu^{2d}$, $K_o = KK_1^{2d}$.

(ii) As a first step in proving $(11.1)'$ we show that, for c small, given any $\sigma \geq 0$ and $a \geq 0$,

$$(11.2)\qquad \int_{a+d}^{a+4d} |e^{\sigma s}u(s)|^2 ds \leq C\int_{a}^{a+d} |e^{\sigma s}u(s)|^2 ds + C\int_{a+4d}^{a+5d} |e^{\sigma s}u(s)|^2 ,$$

here $C \geq 1$ is a fixed constant independent of σ or of the solution u.

In proving (11.2) we may suppose that a = 0; this can be achieved by a translation. Let $\zeta(t)$ be a C^∞ function which equals one on the interval (d,4d), and vanishes outside the interval (0,5d); let c_1 be a bound for $|\frac{d\zeta}{dt}|$. Set $v = e^{\sigma t}\zeta(t)u(t)$, with v = 0 outside the interval (0,5d) and $(L+i\sigma)v = f$. Taking Fourier transforms we find that

$$(\lambda + i\sigma - A)\hat{v}(\lambda) = \hat{f}(\lambda) \ .$$

Arguing as in the proof of Theorem 3.2 we find that

$$\int_{-\infty}^{\infty} |v(t)|^2 dt \leq kM^2 \int_{-\infty}^{\infty} |f(t)|^2 dt \ ;$$

here k is the constant occurring in Lemma 3.1. Since

$$f = \begin{cases} e^{\sigma t}Lu & \text{in the interval } (2d,3d) \\ e^{\sigma t}(\zeta Lu - i\,\frac{d\zeta}{dt}u) & \text{outside the interval} \end{cases}$$

it follows from (8.1) with $\phi \leq c$ that

$$\int_{a+d}^{a+4d} |v(t)|^2 dt \leq kM^2c^2 \int_{a+d}^{a+4d} |v|^2 dt$$
$$+ kM^2(c+c_1)^2\left(\int_{a}^{a+d} |e^{\sigma t}u|^2 dt + \int_{a+4d}^{a+5d} |e^{\sigma t}u|^2 dt\right) \ .$$

Thus if c_o is such that $kM^2c_o^2 = \frac{1}{2}$ and if $c \leq c_o$ the inequality (11.2) follows.

A weaker form of (11.2) is the inequality, for $c \leq c_o$, $\sigma \geq 0$

$$(11.2)' \quad \int_{a+2d}^{a+3d} |u(s)|^2 ds \leq Ce^{-2\sigma d} \int_a^{a+d} |u(s)|^2 ds + Ce^{6\sigma d} \int_{a+4d}^{a+5d} |u|^2 ds.$$

Thus if

$$\int_a^{a+d} |u|^2 ds \geq \int_{a+4d}^{a+5d} |u|^2 ds$$

we may choose $\sigma \geq 0$ so that the terms on the right of the previous inequality are equal, and we find

$$\int_{a+2d}^{a+3d} |u|^2 ds \leq 2C \left(\int_a^{a+d} |u|^2 ds \right)^{\frac{3}{4}} \left(\int_{a+4d}^{a+5d} |u|^2 ds \right)^{\frac{1}{4}} .$$

(iii) Set $g_n = \int_{I_n} |u|^2 ds$, $n = 0,1,\ldots$. According to the preceding we have:

$$(11.3) \quad \begin{aligned} &\text{if} \quad g_{n-1} \geq g_{n+1} \quad \text{then} \quad g_n \leq 2C g_{n-1}^{3/4} \, g_{n+1}^{1/4} , \\ &\text{if} \quad g_{n-1} \leq g_{n+1} \quad \text{then} \quad g_n \leq 2C g_{n+1} , \end{aligned}$$

the last following from (11.2)' for $\sigma = 0$; here $C \geq 1$. These consititute the "convexity" properties.

Let h_n, $n = 0,1,\ldots$ be the solution of the equations

$$h_n = 2C h_{n-1}^{3/4} \, h_{n+1}^{1/4} ,$$

each equation serving to determine h_{n+1}, given h_o and h_1 satisfying

$$h_o = g_o , \quad h_1 \leq g_1 , \quad h_1 \leq h_o .$$

Since $C \geq 1$ the h_n form a decreasing sequence. We claim that

$$g_n \geq h_n , \qquad n = 0,1,\ldots . \tag{11.4}$$

It suffices to prove $g_n \geq h_n'$ for a solution h_n' of the slightly modified system

$$h_n' = 2C' h_{n-1}'^{3/4} \; h_{n+1}'^{1/4} , \quad h_o' = g_o, \quad h_1' < g_1, \quad h_1' \leq h_o' ,$$

with $C' > C$. On letting $C' \to C$, $h_1' \to h_1$ we obtain the desired result.

The proof of the inequalities $g_n \geq h_n'$, $n = 0,1,\ldots$ is rather simple. They are true for $n = 0,1$. Assume that they are false for some value N of n then for some value $n = j$ in the interval $(0,N)$, the ratio g_n/h_n' assumes its maximum $m = g_j/h_j' > 1$. Clearly $0 < j < N$. We have to consider two cases.

If $g_{j-1} \leq g_{j+1}$ then

$$g_j \leq 2Cg_{j+1}$$

and therefore, since the h_n' are decreasing, and $C' > C$,

$$m = \frac{g_j}{h_j'} \leq \frac{C}{C'} \frac{g_{j+1}}{h_{j+1}'} < m , \quad \text{impossible !}$$

If $g_{j-1} \geq g_{j+1}$ we have

$$g_j \leq 2Cg_{j-1}^{3/4} \; g_{j+1}^{1/4}$$

and again we find

$$m = \frac{g_j}{h_j'} \leq \frac{C}{C'}\left(\frac{g_{j-1}}{h_{j-1}'}\right)^{3/4}\left(\frac{g_{j+1}}{h_{j+1}'}\right)^{1/4} < m \ .$$

Thus $g_n \geq h_n'$ for all n, and, as we indicated before (11.4) follows.

(iv) Thus to obtain the lower bound (11.1)' it suffices to obtain the same bound for the h_n:

$$(11.1)'' \qquad h_n \geq KK_1^{t_n} e^{-\beta\mu^{t_n}} \ .$$

Since

$$\frac{h_{n+1}}{h_n^3} = \left(\frac{1}{2C}\right)^4 \frac{h_n}{h_{n-1}^3}$$

it follows that

$$\frac{h_{n+1}}{h_n^3} = \left(\frac{1}{2C}\right)^{4n} \frac{h_1}{h_o^3} \quad ,$$

or

$$h_{n+1}(2C)^{-2(n+1)}\kappa = (h_n(2C)^{-2n}\kappa)^3$$

where $\kappa = \frac{1}{2C}(h_1/h_o^3)^{1/2}$. Consequently,

$$h_n(2C)^{-2(n+1)}\kappa = (h_o\kappa)^{3^n} = \left(\frac{1}{2C}(h_1/h_o)^{1/2}\right)^{3^{t_n/2d}}$$

or

$$h_n = \frac{(2C)^2}{\kappa}(2C)^{t_n/d}\left(\frac{1}{2C}\sqrt{\frac{h_1}{h_o}}\right)^{3^{t_n/2d}} \ .$$

This is of the form (11.1)" with $K = \frac{(2C)^2}{\kappa}$, $K_1 = (2C)^{1/d}$, $\mu = 3^{1/2d}$, and $-\beta = \log\left(\frac{1}{2C}\sqrt{\frac{h_1}{h_o}}\right)$. Q.E.D.

Chapter IV

Regularity of Solutions

12. Differentiability and analyticity

We deal with functions $u(t)$ defined in some interval of the reals with values in a Banach space Y, with norm $|\ |$. (In this chapter we shall suppose, for convenience, that $X = Y$ although many of the results are easily extended to the more general case.) We consider the equation (2)

$$(12.1) \qquad Lu = \frac{1}{i}\frac{du}{dt} - Au = f(t)$$

where A is a closed operator in Y and where u is differentiable in some sense, takes its values in D_A (domain of definition of A) and satisfies (12.1). We shall first give some necessary conditions in order that a solution of (12.1) with f belonging to some class of functions over an interval, will also belong to the same class in some interior interval. We shall consider here the classes C^n, C^∞ and the class of analytic functions. We note however that various other classes of functions could also be treated by the same procedure which consists in applying the closed graph theorem in a suitable form, deriving a priori estimates and applying these to exponential functions. Afterwards we consider sufficient conditions.

Denote by $C^n[-a,a]$, the Banach space of n times continuously differentiable functions in $|t| \leq a$ with values in Y, and with the usual maximum norm:

$$|u|_n = \max_{0 \le j \le n} \max_{|t| \le a} \left| \frac{d^j u(t)}{dt^j} \right| .$$

The first necessity result for differentiability is the following

<u>Theorem 4.1</u>: <u>Suppose that solutions of</u> (12.1) <u>possess the following property</u>: <u>there exist numbers</u> $0 < a' < a$ <u>and integers</u> $k \ge 2$, $s \ge -1$, <u>such that if</u> $u(t) \in C^1[-a,a]$ <u>and</u> $f = Lu \in C^{k+s}[-a,a]$, <u>then</u> $u \in C^k[-a',a']$. <u>Then, the following inequality must hold</u>:

$$(12.2) \qquad |(A-\lambda)\phi| \ge C|\lambda|^{-s} e^{-(a-a')|\text{Im}\,\lambda|} |\phi|$$

<u>for all</u> $\phi \in D_A$ <u>and all complex numbers</u> λ <u>such that</u>

$$(12.2)' \qquad |\text{Im}\,\lambda| \le \frac{k-1}{a-a'} \log |\text{Re}\,\lambda| - C_1 , \qquad |\lambda| \ge N_o .$$

<u>Here</u> C, N_o <u>and</u> C_1 <u>are certain positive constants</u>.

<u>Proof</u>: Consider the graph G: (u,Lu) for all $u \in C^1[-a,a]$ with values in D_A such that $Lu \in C^{k+s}[-a,a]$. It is readily seen that G is a closed linear set in the product space $C^1[-a,a] \times C^{k+s}[-a,a]$. Thus, with the induced topology G is a Banach space. Consider the mapping $(u,Lu) \longrightarrow u$ from G into $C^k[-a',a']$. By our assumption this is everywhere defined linear transformation and it is readily seen to be closed. Hence, by the closed graph theorem the mapping is continuous. In other words, denoting by $|u|_n'$ the norm in $C^n[-a',a']$, we must have:

$$(12.3) \qquad |u|_k' \le K(|Lu|_{k+s} + |u|_1)$$

for all admissible functions u with $(u,Lu) \in G$ and some constant K.

Now, apply (12.3) to functions of the form: $u(t) = \phi e^{i\lambda t}$ where ϕ is an element in D_A and $\lambda = \mu + i\nu$ a complex number. A simple computation shows that with another constant K_1 (independent of λ or ϕ) one must have for $|\lambda| \geq 1$:

$$|\lambda|^k |\phi| e^{a'|\nu|} \leq K_1(|(A-\lambda)\phi| \; |\lambda|^{k+s} + |\lambda| \; |\phi|)e^{a|\nu|} .$$

Hence

$$(12.4) \qquad |\lambda|^k e^{(a'-a)|\nu|} \leq K_1\left(\frac{|(A-\lambda)\phi|}{|\phi|} |\lambda|^{k+s} + |\lambda|\right) .$$

We now restrict λ by the relation:

$$|\lambda|^{k-1} e^{(a'-a)|\mathrm{Im}\ \lambda|} \geq 2K_1$$

or

$$(12.5) \qquad |\mathrm{Im}\ \lambda| \leq \frac{k-1}{a-a'} \log |\lambda| - C_1 \qquad \left(C_1 = \frac{\log (2K_1)}{a-a'}\right) .$$

Then, for such values of λ one obtains

$$|\lambda|^k e^{(a'-a)|\mathrm{Im}\ \lambda|} \leq 2K_1 \frac{|(A-\lambda)\phi|}{|\phi|} |\lambda|^{k+s}$$

or

$$(12.6) \qquad |(A-\lambda)\phi| \geq C|\lambda|^{-s} e^{-(a-a')|\mathrm{Im}\ \lambda|} |\phi| .$$

This proves the theorem (since (12.2)' implies (12.5).

<u>Remark</u>: Suppose in addition that the resolvent $R(\lambda;A) = (\lambda I-A)^{-1}$ exists at some point in each component of the region (12.2)'. It follows readily from (12.6) that the resolvent exists in the region (12.2)' and that in this region

$$(12.7) \qquad |R(\lambda;A)| \leq \text{constant } |\lambda|^s e^{(a-a')|\mathrm{Im}\ \lambda|} .$$

In the special case $s = -1$ we have for real λ, $|\lambda| \geq N_o$: $|R(\lambda;A)| = O(\frac{1}{\lambda})$. This implies that the resolvent exists in a double sector $|\arg (\pm\lambda)| \leq \alpha$, $0 < \alpha < \frac{\pi}{\alpha}$, $(|\lambda| \geq \Lambda_o)$, where it satisfies

$$|R(\lambda;A)| = O(\frac{1}{\lambda}) .$$

Suppose, now, that A has the property that if $u \in C^1[-a,a]$ and $Lu \in C^\infty[-a,a]$ then $u \in C^\infty[-a',a']$. To obtain a necessary condition in this situation one modifies the previous argument. Consider again the graph $G = \{(u,Lu) : u \in C^1[-a,a], Lu \in C^\infty[-a,a]\}$ which is a closed linear set in the product space $C^1[-a,a] \times C^\infty[-a,a]$. The map $(u,Lu) \longrightarrow u$ from G (with the induced topology) into $C^\infty[-a',a']$ is everywhere defined by our assumption and is readily seen to be closed. Hence, by the closed graph theorem it is continuous. Now the topology in $C^\infty[-a,a]$ is given by the sequence of norms $|u|_k$ in $C^k[-a,a]$. Hence, continuity of the above mapping means that for every $k \geq 2$ there should exist a position integer s for which (12.3) holds. Summing up we obtain

<u>Theorem 4.1'</u>: <u>A necessary condition that</u> $u \in C^1[-a,a]$, $Lu \in C^\infty[-a,a]$ <u>would imply</u> $u \in C^\infty[-a',a']$, <u>is that for every integer</u> $k \geq 2$ <u>there should exist an integer</u> s <u>such that the inequality</u> (12.2) <u>should hold in some "logarithmic" region</u> (12.2)'.

Finally, we shall give also necessary conditions for analyticity. We have

<u>Theorem 4.2</u>: <u>A necessary condition that every function</u> $u \in C^1[-a,a]$ <u>with values in</u> D_A <u>and such that</u> Lu <u>is analytic in</u> $|t| \leq a$ <u>be itself</u>

<u>analytic in an interior interval</u> $|t| \leq a'$ $(0 < a' < a,\ a'$ <u>fixed</u>) <u>is that there exist numbers</u> $0 < \alpha < \frac{\pi}{2}$, $b \geq 0$ <u>and positive constants</u> C, N_o <u>such that</u>

$$|(A-\lambda)\phi| \geq Ce^{-b|\lambda|}|\phi| \tag{12.8}$$

<u>for all</u> λ <u>such that</u> $|\arg(\pm\lambda)| \leq \alpha$, $|\lambda| \geq N_o$ <u>and</u> $\phi \in D_A$.

<u>Proof</u>: Denote by M_η (η a given positive number) the class of bounded analytic functions with values in Y defined in the rectangle $|\mathrm{Re}\ t| < a+\eta$, $|\mathrm{Im}\ t| < \eta$ in the complex t plane. M_η is a Banach space if one chooses as a norm:

$$\|u\|_\eta = \sup_{\substack{|\mathrm{Re}\ t|<a+\eta \\ |\mathrm{Im}\ t|<\eta}} |u(t)| . \tag{12.9}$$

Similarly denote by M_η' the Banach space of bounded analytic functions in the rectangle $|\mathrm{Re}\ t| < a'+\eta$, $|\mathrm{Im}\ t| < \eta$. We denote by $\|u\|_\eta'$ the corresponding norm in M_η'.

Assume that the analyticity property of Theorem 4.2 holds. We shall show that (12.8) must be satisfied in a double angle. To this end let $\eta_o > 0$ be fixed and consider all functions $u \in C^1[-a,a]$ with values in D_A such that $Lu \in M_{\eta_o}$. (Clearly this set contains many functions, e.g. all functions of the form $u = \phi g(t)$ with $\phi \in D_A$ and $g(t)$ a scalar entire function.) Consider the graph $G = \{(u,Lu)\}$ of all such functions. It is readily seen that G is a closed linear subset of $C^1[-a,a] \times M_{\eta_o}$. Thus, with the induced topology G is a non-trivial Banach space. By assumption it follows that if $(u,Lu) \in G$ then u is analytic in $|t| \leq a'$. Let

$G_n = \{(u,Lu)\in G,\ u\in M'_{1/n}\}$. Clearly $\bigcup_{n=1}^{\infty} G_n = G$. Since G is a set of the second category, it follows that one of the G_n's, say G_{n_1}, is also a set of the second category. Set $\eta_1 = 1/n_1$. Consider the mapping from G_{n_1} into M'_{η_1} defined by: $(u,Lu) \to u$. This is an everywhere defined linear transformation on G_{n_1} which is closed (in G_{n_1}). Hence by one form of the closed graph theorem it follows that transformation is bounded on G_{n_1} and thus can be extended by continuity to $\overline{G}_{n_1}$. Now, since G_{n_1} is a linear set of the second category in the Banach space G it follows that G_{n_1} is dense in G so that by the definition of G_{n_1} we actually have: $G_{n_1} = \overline{G}_{n_1} = G$.

The above considerations yield the following estimates for $(u,Lu)\in G$:

$$\|u\|'_{\eta_1} \le K\left(\|Lu\|_{\eta_o} + |u|_1\right) \tag{12.10}$$

where K is some constant. Taking $u(t) = \phi e^{i\lambda t}$ with $\phi\in D_A$ and $\lambda = \mu + i\nu$, $t = s + ir$, it is readily seen that

$$\|e^{i\lambda t}\phi\|_\eta = \max_{\substack{|s|\le a+\eta\\ |r|\le \eta}} e^{-\mu r-\nu s}|\phi| = e^{|\mu|\eta+|\nu|(a+\eta)}|\phi| .$$

Hence, (12.10) yields (for $|\lambda| \ge 1$):

$$|\phi| e^{|\mu|\eta_1+|\nu|(a'+\eta_1)} \le K\left(|(A-\lambda)\phi| e^{|\mu|\eta_o+|\nu|(a+\eta_o)} + |\phi|\ |\lambda| e^{|\nu|a}\right). \tag{12.11}$$

It is now readily seen that if

(12.12) $\quad |\nu| \le \frac{\eta_1}{a-a'} |\mu|$ and $|\lambda| \ge N_0$, sufficiently large ,

then:

$$e^{|\mu|\eta_1+|\nu|(a'+\eta_1)} \ge 2K|\lambda|e^{|\nu|a} .$$

Hence, if (12.12) holds we obtain from (12.11):

$$|\phi|e^{|\mu|\eta_1+|\nu|(a'+\eta_1)} \le 2K|(A-\lambda)\phi|e^{|\mu|\eta_0+|\nu|(a+\eta_0)} .$$

Or, in the double angle $|\arg(\pm\lambda)| \le \alpha$, $|\lambda| \ge N_0$ where $\tan\alpha = \frac{\eta_1}{a-a'}$, we have

$$|(A-\lambda)\phi| \ge \frac{1}{2K} e^{-(|\mu|+|\nu|)(\eta_0-\eta_1)-|\nu|(a-a')}|\phi| .$$

This gives (12.8) and establishes the theorem.

<u>Remark</u>: As before we conclude from (12.8) that if the resolvent $R(\lambda;A)$ exists for a sufficiently large positive number λ^+ and also for a sufficiently large negative number λ^-, then a necessary condition for solutions of (12.1) to possess the analyticity property of Theorem 4.2 is that the resolvent should exist in some double angle $|\arg(\pm\lambda)| \le \alpha$ $(0 < \alpha < \frac{\pi}{2})$, $|\lambda| \ge N_0$, and grow at most exponentially in the double angle as $\lambda \to \infty$.

We pass now to the problem of giving sufficient conditions for differentiability. We shall show in the following that the necessary conditions derived above are almost sufficient for the various differentiability theorems. More precisely, the necessary conditions somewhat strengthened will be shown to be sufficient.

For a treatment of the differentiability of solutions of $Lu = 0$ in case A generates a semigroup see K. Yosida [2] where also other references are given.

For simplicity we shall assume in the following that the solutions of (12.1) are taken in the most restricted pointwise sense and are C^1 functions. However, it will be clear from the proof of the various differentiability results (all of which employ suitable integral representation formulas) that these results hold under much weaker assumptions on the solution $u(t)$ which need not be differentiable or even continuous but only a weak solution of (12.1) in some sense. Indeed, if one considers such generalized solutions the corresponding results **yield** regularity theorems showing that weak solutions are actually smooth functions.

Theorem 4.3: *Suppose that the resolvent* $R(\lambda;A)$ *exists in a region*

$$(12.13) \qquad \mathcal{D}: \quad |\mathrm{Im}\ \lambda| \leq \frac{\log |\mathrm{Re}\ \lambda|}{c}, \qquad |\lambda| \geq N_0 ,$$

where c *and* N_0 *are some positive numbers and that*

$$(12.13)' \qquad |R(\lambda;A)| \leq \text{constant } |\lambda|^s e^{\Delta|\mathrm{Im}\ \lambda|} \quad \text{in } \mathcal{D},$$

where $s \geq -1$ *and* $\Delta > 0$ *are constants. Let* $u(t)$ *be a* C^1 *solution of* (12.1) *in the interval* $|t| < a$. *Suppose that in the same interval* $f \in C^{k+[2+s]}$; k *being some integer* ≥ 2. *Set*:

$$a' = a-\Delta-c(s+k+1)$$

and suppose that $a' > 0$. Then $u \in C^k$ in the subinterval $|t| < a'$.

An immediate consequence of Theorem 4.3 is the following result on infinite differentiability.

Theorem 4.3': Suppose that A has the following property: For every $\varepsilon > 0$ there exists a number $N_0(\varepsilon) > 0$ such that $R(\lambda;A)$ exists in the domain:

$$\mathcal{D}_\varepsilon: \qquad |\mathrm{Im}\ \lambda| \le \frac{\log |\mathrm{Re}\ \lambda|}{\varepsilon}, \qquad |\lambda| \ge N_0(\varepsilon),$$

and

$$|R(\lambda:A)| \le C_\varepsilon |\lambda|^s e^{\Delta|\mathrm{Im}\ \lambda|} \quad \text{in} \quad \mathcal{D}_\varepsilon$$

where $s \ge -1$ and $\Delta \ge 0$ are constants independent of ε whereas C_ε depends on ε. Then every C^1 solution of (12.1) in $|t| < a$ with $f \in C^\infty$ in $|t| < a$ is infinitely differentiable in the subinterval $|t| < a-\Delta$.

Proof of Theorem 4.3: Let a'' be an arbitrary number such that $0 < a'' < a'$. It will suffice to establish the differentiability property of u in $|t| < a''$. With $\delta = a' - a''$, and $\zeta(t)$ a scalar C^∞ function on the real line such that $\zeta(t) \equiv 1$ for $|t| \le a-\delta$, $\zeta(t) \equiv 0$ for $|t| \ge a - \frac{\delta}{2}$, set $v = \zeta u$ for $|t| < a$, $v \equiv 0$ for $|t| \ge a$. Then v is a C^1 function on the line such that $v \equiv u$ for $|t| \le a-\delta$ and:

$$(12.14) \qquad Lv = \zeta f + i\zeta' u, \qquad \text{for} \quad |t| < a.$$

We define $f_0 = \zeta f$ for $|t| \le a$, $f_0 \equiv 0$ for $|t| > a$; $g_+ = i\zeta' u$ for $-a \le t \le -a+\delta$, $g_+(t) \equiv 0$ otherwise; $g_- = i\zeta' u$ for $a-\delta \le t \le a$,

$g_- \equiv 0$ otherwise. Clearly with these definitions we have on the whole line:

(12.14)' $$Lv = f_o + g_+ + g_- .$$

Taking Fourier transforms we find that

$$(\lambda - A)\hat{v}(\lambda) = \hat{f}_o(\lambda) + \hat{g}_+(\lambda) + \hat{g}_-(\lambda) .$$

Since v, f_o, g_+ and g_- are of compact support the corresponding Fourier transforms are entire functions of exponential type (with values in Y) so that the preceding equation actually holds in the entire complex λ plane. Thus, whenever the resolvent $R(\lambda) = R(\lambda;A)$ exists:

$$\hat{v}(\lambda) = R(\lambda)\hat{f}_o(\lambda) + R(\lambda)\hat{g}_+(\lambda) + R(\lambda)\hat{g}_-(\lambda) ,$$

so that, for $|t| \leq a'' < a-\delta$

$$\sqrt{2\pi}\, u(t) = \int_{-N_o}^{N_o} e^{i\lambda t}\hat{v}(\lambda) + \int_{-\infty}^{-N_o} e^{i\lambda t}R(\lambda)(\hat{f}_o + \hat{g}_+ + \hat{g}_-)d\lambda$$
$$+ \int_{N_o}^{\infty} e^{i\lambda t}R(\lambda)(\hat{f}_o + \hat{g}_+ + \hat{g}_-)d\lambda .$$

We claim that after suitable deformations of contours u(t) can be expressed in the following form for $|t| \leq a''$

(12.15) $$\sqrt{2\pi}\, u(t) = u_o(t) + u_1(t) + u_2(t) + u_1^+(t) + u_2^+(t) + u_1^-(t) + u_2^-(t)$$

where these terms are absolutely convergent integrals:

$$u_o(t) = \int_{-N_o}^{N_o} e^{i\lambda t}\hat{v}(\lambda)d\lambda ,$$

$$u_1 = \int_{-\infty}^{-N_o} e^{i\lambda t}R(\lambda)\hat{f}_o d\lambda , \qquad u_2 = \int_{N_o}^{\infty} e^{i\lambda t}R(\lambda)\hat{f}_o d\lambda ,$$

$$u_j^{\pm}(t) = \int_{\Gamma_j^{\pm}} e^{i\lambda t}R(\lambda)\hat{g}_{\pm}(\lambda)d\lambda , \qquad j = 1,2,$$

here Γ_1^+ is the infinite curve in the first quadrant of the $\lambda = \mu + i\nu$ plane given by

$$(12.16) \qquad \Gamma_1^+ : \quad c\nu = \log \mu - \log N_o \qquad \text{for} \quad \mu \geq N_o$$

(c being the constant in (12.13)), Γ_2^+ is the reflected image of Γ_1^+ in the imaginary axis, and Γ_1^-, Γ_2^- are the reflections of Γ_1^+, Γ_2^+ in the real axis, all curves being oriented with increasing Re λ.

Suppose for the moment that these integrals have been shown to be absolutely convergent. Then this representation for u(t), i.e. the deformation of contour, is achieved with the aid of the "multiplier" function (7.6) employed just as in §7: one deforms the contours for the functions $v_\varepsilon(t)$ given by (7.6)", so that $\hat{v}(\lambda)$ is replaced by $q(\varepsilon\lambda)\hat{v}(\lambda)$, and then lets $\varepsilon \to 0$.

We shall now show for $|t| < a''$ that not only does each of the last seven integrals converge absolutely, but even after formal differentiation $j \leq k$ times, the resulting integrals

converge absolutely and uniformly in every closed subinterval of $|t| < a''$. From this it would follow readily that $u \in C^k$ in $|t| < a''$ and the proof will be complete.

The above remark clearly holds for $u_o(t)$ which is (after extension) an entire function in the complex t plane. Consider next $u_1(t)$ and $u_2(t)$. Since $f_o(t)$ is of compact support and of class $C^{k+[2+s]}$ it follows from (12.13)' that

$$|\lambda|^k |R(\lambda)\hat{f}_o(\lambda)| = O(|\lambda|^{s-[2+s]}) \quad \text{for real} \quad \lambda \longrightarrow \pm\infty .$$

Hence, since $s-[2+s] < -1$

$$u_2(t) = \int_{N_o}^{\infty} e^{i\lambda t} R(\lambda)\hat{f}_o(\lambda)d\lambda$$

and, similarly $u_1(t)$, belong to C^k on the whole real line.

Consider now

$$u_1^+(t) = \int_{\Gamma_1^+} e^{i\lambda t} R(\lambda)\hat{g}_+(\lambda)d\lambda .$$

Since the support of $g_+(t)$ is contained in $-a < t \leq -a+\delta$ it follows that $\hat{g}_+(\lambda)$ (which is an entire function of exponential type) satisfies:

$$|e^{-ia\lambda}\hat{g}_+(\lambda)| = O(e^{\delta|\text{Im}\ \lambda|})$$

in the whole plane. It follows, with the aid of (12.13)' that on Γ_1^+:

$$\begin{aligned}|\lambda^k e^{i\lambda t} R(\lambda)\hat{g}_+(\lambda)| &= |\lambda^k e^{i\lambda(t+a)} R(\lambda)(e^{-ia\lambda}\hat{g}_+(\lambda))| \\ &\leq \text{constant } \mu^k e^{-\nu(t+a)} \mu^s e^{(\Delta+\delta)\nu} \\ &\leq \text{constant } \mu^{k+s+\frac{\Delta+\delta-(t+a)}{c}}, \text{ using (12.16).}\end{aligned}$$

The k times formally differentiated integral $u_1^+(t)$ will thus converge absolutely and uniformly in some closed interval, if throughout the interval:

$$k + s + \frac{\Delta+\delta-(t+a)}{c} < -1$$

or

$$t > -a + \Delta + \delta + c(k+s+1) = -a'+\delta = -a'' .$$

Thus, $u_1^+(t) \in C^k$ for $t > -a''$. Similarly one establishes the desired absolute convergence of the other k times differentiated integrals. It follows that $u_2^+(t) \in C^k$ for $t > -a''$; while $u_1^-(t)$ and $u_2^-(t)$ belong to C^k for $t < a''$. Combining these results it follows that $u \in C^k$ for $|t| < a'-\delta$ (and consequently for $|t| < a'$) and the proof is complete.

<u>Remark</u>: Suppose that the Banach space is a Hilbert space. Let A be an operator whose resolvent satisfies the conditions of Theorem 4.3 with s an <u>integer</u>. Let u now be a solution of $Lu = f$ in $|t| < a$ and suppose that $f(t)$ has k+s derivatives in L_2 in the interval. <u>Conclusion</u>: u possesses k derivatives in L_2 in any closed subinterval of $|t| < a'$. The proof in this case is very much the same except that Plancheral's theorem is used in treating

$u_1(t)$ and $u_2(t)$. One finds that these functions possess k derivatives in L_2 on the whole line. The conditions in this L_2 result are closer to the necessity condition than are the conditions in the general theorem just proved.

Finally, we give sufficient conditions for analyticity.

Theorem 4.4: Suppose that $R(\lambda;A)$ exists in the double infinite sector:

$$\sum : \quad |\arg(\pm\lambda)| \leq \alpha , \qquad |\lambda| \geq N_0 ,$$

$0 < \alpha < \frac{\pi}{2}$ and that:

$$(12.17) \qquad |R(\lambda;A)| = O(e^{\Delta|\mathrm{Im}\,\lambda|+\varepsilon|\lambda|}) \qquad \text{as} \quad \lambda \longrightarrow \infty \quad \text{in} \sum$$

for every $\varepsilon > 0$ and some $\Delta \geq 0$. Let $u(t)$ be a C^1 function in the interval $|t| < a$ with values in D_A and suppose that $Lu = f$ is analytic for $|t| < a$. Then u is analytic in the subinterval $|t| < a-\Delta$.

For related results in the case that A generates a semigroup see K. Yosida [1]. The regularity theorems here should of course be extended to equations in which A is permitted to depend, say analytically, on t. In case, for each t, A(t) is the generator of a semigroup then, under suitable conditions, the analyticity of solutions of $Lu = 0$ has been shown by H. Komatsu [1], where other references, as well as applications to parabolic equations in a cylinder are also given.

Proof: The proof of the theorem is similar to that of Theorem 4.3. Let us first note that without loss of generality we may assume

that u is continuous for $|t| \leq a$ and f is analytic for $|t| \leq a$. We shall actually establish a more general result than the one stated. Namely, we shall assume that $R(\lambda;A)$ is of some exponential growth $O(e^{b|\lambda|})$ in $\sum$ and satisfies the weaker growth relations

$$(12.18) \qquad |R(\lambda;A)| = O(e^{\Delta'|\text{Im }\lambda|}) \qquad \text{for } \arg\lambda = \pm\alpha\,, \quad |\lambda| \to \infty\,.$$

and

$$(12.18)' \qquad |R(\lambda;A)| = O(e^{\Delta''|\lambda|}) \qquad \text{for real } \lambda \to \pm\infty\,,$$

where $\Delta' \geq 0$, $\Delta'' \geq 0$ are some constants. Under the assumptions (12.18), (12.18)' we shall show that if $f(t)$ is analytic in the rectangle $|\text{Re } t| \leq a$, $|\text{Im } t| \leq \eta$ in the complex t-plane with $\eta > \Delta''$, then u is analytic in the subinterval $|t| < a-\Delta' - \frac{\eta}{\sin\alpha}$ of the real t axis. Clearly the theorem will follow by taking $\Delta' = \Delta+\varepsilon$, $\Delta'' = \varepsilon$ with $\varepsilon > 0$ arbitrarily small.

To prove the last assertion we start by extending $u(t)$ and $f(t)$ to the whole real axis by setting $u = 0$ and $f \equiv 0$ for $|t| > a$. Let $\hat{u}$ and $\hat{f}$ be the corresponding Fourier transforms, so that

$$\hat{u}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} u(t)e^{-it\lambda}dt\,.$$

Since $Lu = f$ we find that

$$(\lambda-A)\hat{u}(\lambda) = \frac{1}{i\sqrt{2\pi}}\, u(-a)e^{ia\lambda} - \frac{1}{i\sqrt{2\pi}}\, u(a)e^{-ia\lambda} + \hat{f}(\lambda)$$

so that in $\sum$:

$$(12.19) \qquad \hat{u}(\lambda) = \frac{e^{ia\lambda}}{i\sqrt{2\pi}} R(\lambda)(u(-a)) - \frac{e^{-ia\lambda}}{i\sqrt{2\pi}} R(\lambda)u(a) + R(\lambda)\hat{f}(\lambda) .$$

Since $f(t)$ can be extended as an analytic function of $t = s+ir$ into the rectangle $|s| \leq a$, $|r| \leq \eta$ we may write $\hat{f}(\lambda)$ as

$$\sqrt{2\pi}\, \hat{f}(\lambda) = \int_{-a}^{a} e^{-is\lambda} f(s)ds$$

$$(12.20) \qquad = e^{\eta\lambda} \int_{-a}^{a} e^{-is\lambda} f(s+i\eta)ds + ie^{ia\lambda} \int_{0}^{\eta} e^{r\lambda} f(-a+ir)dr$$

$$- ie^{-ia\lambda} \int_{0}^{\eta} e^{r\lambda} f(a+ir)dr .$$

Similarly we find that (12.20) holds also when η is replaced by $-\eta$. These results may be expressed in the form

$$\sqrt{2\pi}\, \hat{f}(\lambda) = e^{\eta\lambda} g_2(\lambda) + e^{ia\lambda} h_2^+(\lambda) + e^{-ia\lambda} h_2^-(\lambda)$$

$$= e^{-\eta\lambda} g_1(\lambda) + e^{ia\lambda} h_1^+(\lambda) + e^{-ia\lambda} h_1^-(\lambda)$$

where g_1, g_2, $h_1^\pm$, $h_2^\pm$ are entire functions of exponential type such that g_1, g_2 are bounded on the real axis, and

$$h_j^\pm(\lambda) = O(e^{\eta|\lambda|}) , \qquad j = 1,2$$

in the complex plane.

Taking inverse Fourier transforms we have, for $|t| < a-\Delta-\eta/\sin\alpha$, on substituting the above two expressions for $\hat{f}$ into (12.19) for $t < -N_0$ and $t > N_0$ respectively,

$$(12.21)\quad \begin{aligned} 2\pi\ u(t) = \sqrt{2\pi}\int_{-N_o}^{N_o} & e^{i\lambda t}\hat{u}(\lambda)d\lambda \\ & + \int_{N_o}^{\infty} e^{i\lambda t-\eta\lambda}R(\lambda)g_1 d\lambda + \int_{-\infty}^{-N_o} e^{i\lambda t+\eta\lambda}Rg_2 d\lambda \\ & + \int_{N_o}^{\infty} e^{i\lambda(t+a)}Rk_1^+(\lambda)d\lambda + \int_{N_o}^{\infty} e^{i\lambda(t-a)}Rk_1^-(\lambda)d\lambda \\ & + \int_{-\infty}^{-N_o} e^{i\lambda(t+a)}Rk_2^+(\lambda)d\lambda + \int_{-\infty}^{-N_o} e^{i\lambda(t-a)}Rk_2^-(\lambda)d\lambda \end{aligned}$$

where we have set

$$k_j^+(\lambda) = -iu(-a)+h_j^+(\lambda)\ ,\quad k_j^-(\lambda) = iu(a)+h_j^-(a)\ ,\quad j = 1,2\ .$$

We note that $k_j^{\pm}(\lambda) = O(e^{\eta|\lambda|})$ in the plane.

We wish now to deform the lines of integration in the last four integrals to lines on the boundary of $\sum$ so that the expression u(t) becomes

$$(12.21)'\quad 2\pi\ u(t) = u_o+u_1+u_2+u_1^++u_1^-+u_2^++u_2^-$$

where u_o, u_1, u_2 represent the first three integrals on the right of (12.20), and

$$u_1^+ = \int_{N_o}^{N_o+e^{i\alpha}\cdot\infty} e^{i\lambda(t+a)}Rk_1^+ d\lambda\ ,\quad u_1^- = \int_{N_o}^{N_o+e^{-i\alpha}\cdot\infty} e^{i\lambda(t-a)}Rk_1^- d\lambda$$

$$u_2^+ = \int_{-N_o+e^{i(\pi-\alpha)}\cdot\infty}^{-N_o} e^{i\lambda(t+a)}Rk_2^+ d\lambda,\quad u_2^- = \int_{-N_o+e^{i(\alpha-\pi)}\cdot\infty}^{-N_o} e^{i\lambda(t-a)}Rk_2^- d\lambda\ .$$

These contour deformations are justified as before with the aid of the multiplier (7.6) (with the value of r in the multiplier chosen so that $1 < r < \frac{\pi}{2\alpha}$) provided that the integrals $u_j^{\pm}$ converge absolutely.

Now we shall show that for $|t| < a-\Delta-\eta/\sin\alpha$ the integrals not only converge absolutely but are also analytic in t, giving the desired result. Clearly the integral u_0 converges absolutely and represents an entire function. Since $|g_1|$ is bounded on the real axis, while $R(\lambda)$ satisfies (12.18)' we see that the integral u_1 converges absolutely and represents an analytic function in the complex $t = s + ir$ plane for $r > -(\eta-\Delta'')$; recall that $\eta > \Delta''$. Similarly u_2 is analytic in the half plane $r < \eta-\Delta''$. In treating the other four integrals we use the following estimates on the corresponding rays:

$$|R(\lambda)k_j^{\pm}(\lambda)| = O(e^{\Delta'|\mathrm{Im}\ \lambda|+\eta|\lambda|}) , \quad j = 1,2 .$$

From this it follows readily that u_1^{+} converges absolutely and is analytic in the half plane

$$s \sin\alpha + r \cos\alpha > -d$$

where

$$d = (a-\Delta')\sin\alpha - \eta .$$

Similarly we find that the remaining integrals converge absolutely and represent analytic functions in the half planes:

$$u_1^- \text{ in } -s \sin\alpha + r\cos\alpha > -d$$

$$u_2^+ \text{ in } s \sin\alpha - r\cos\alpha > -d$$

$$u_2^- \text{ in } -s \sin\alpha - r\cos\alpha > -d\ .$$

In particular, then, for real t we see that the u_j^+ are analytic for $t > -(a-\Delta')+\eta/\sin\alpha$ while the u_j^- are analytic for $t < a-\Delta'-\eta/\sin\alpha$. Combining these results we find that all the integrals, and consequently u itself, are analytic on the real segment $|t| < a-\Delta'-\eta/\sin\alpha$, completing the proof.

Remarks: 1) Let P be a linear operator mapping the domain of A into Y. One may be interested in the analyticity of Pu in case f is analytic. It is clear from the proof of Theorem 4.4 that if we replace $R(\lambda;A)$ by $PR(\lambda;A)$, and assume that $R(\lambda)$ exists for λ real, $|\lambda| \geq N_o$, then Pu is analytic in the subinterval $|t| < a-\Delta$.

Similar extensions of remarks 2) and 3) also hold.

2) If we consider solutions of the homogeneous equation $Lu = 0$ the representation formula is simplified considerably; we have

$$(12.22)\quad 2\pi\, u(t) = \sqrt{2\pi}\int_{-N_o}^{N_o} e^{i\lambda t}\hat{u}(\lambda)d\lambda$$

$$- i\int_{N_o}^{N_o+e^{i\alpha}\cdot\infty} e^{i\lambda(t+a)}R(\lambda)u(-a)d\lambda + i\int_{N_o}^{N_o+e^{-ia}\cdot\infty} e^{i\lambda(t-a)}R(\lambda)u(a)d\lambda$$

$$- i\int_{-N_o+e^{i(\pi-\alpha)}\cdot\infty}^{-N_o} e^{i\lambda(t+a)}R(\lambda)u(-a)d\lambda + i\int_{-N_o+e^{i(\alpha-\pi)}\cdot\infty}^{-N_o} e^{i\lambda(t-a)}R(\lambda)u(a)d\lambda$$

Formula (12.22) holds under the assumption that $R(\lambda)$ exists in the double angle $|\arg(\pm\lambda)| \leq \alpha$ for $|\lambda| \geq N_o$ and satisfies there $|R(\lambda)| = O(e^{|\lambda|^\rho})$ for some $\rho < \frac{\pi}{2\alpha}$ and, moreover, $|R(\lambda)| = O(e^{\Delta|\mathrm{Im}\,\lambda|})$ on the sides of the angles. Under these conditions it follows readily from (12.22) that every solution of $Lu = 0$ is analytic in the rhombus in the complex t plane having the real segment $-(a-\Delta') \leq t \leq a-\Delta'$ as a diagonal and 2α as the angle at the two vertices $\pm(a-\Delta')$.

3) The assumption on the resolvent of <u>symmetric</u> angles with respect to the real axis was made for the sake of convenience. The results are easily generalized to non-symmetric angles. (See Remark 1 after Theorem 2.8.) If one considers solutions of $Lu = 0$ on a half-line which grow at most exponentially the assumptions could be relaxed even more. We mention one such result which follows easily by the method of proof of Theorem 4.4.

<u>Theorem 4.5</u>: <u>Suppose that</u> $R(\lambda;A)$ <u>exists in the sector</u> $0 \leq \arg\lambda \leq \alpha < \pi$, $|\lambda| \geq N_1$ <u>where it satisfies</u>: $|R(\lambda)| = O(e^{|\lambda|^{\rho_1}})$ <u>with</u> $\rho_1 < \frac{\pi}{\alpha}$; <u>suppose also that</u> $R(\lambda;A)$ <u>exists in</u> $\pi-\beta \leq \arg\lambda \leq \pi$, <u>if</u> $|\lambda| \geq N_2$ $(\beta \leq \pi-\alpha)$ <u>and satisfies there</u> $R(\lambda) = O(e^{|\lambda|^{\rho_2}})$ <u>with</u> $\rho_2 < \frac{\pi}{\beta}$. <u>Suppose, moreover, that</u>

$$|R(\lambda)| = O(e^{\Delta\,\mathrm{Im}\,\lambda}) \qquad \underline{\text{for}}\quad \arg\lambda = \alpha$$
$$\underline{\text{and}}\quad \arg\lambda = \pi-\beta\ .$$

<u>Then, every solution of</u> $Lu = 0$ <u>which exists and belongs to</u> L_1 <u>on the half-line</u> $\gamma \geq 0$ <u>can be extended analytically into the complex</u> t-<u>plane in the angle</u>: $-\alpha < \arg(t-\Delta) < \beta$.

Theorem 4.5 may be considered as an improvement of Theorem 2.3.

4) Suppose that u is a solution of the homogeneous equation $Lu = 0$ on the interval $-a \leq t \leq a$. It is of interest to see whether there exists a solution with the <u>same</u> initial value of a slightly perturbed equation (recall that the initial value problem is not necessarily well posed). We present a result in this direction which follows immediately from Remark 2), in particular from the representation (12.22).

<u>Theorem 4.6</u>: <u>Assuming the preceding, suppose that</u> $R(\lambda)$ <u>exists in the double angle</u> $|\arg(\pm\lambda)| \leq \alpha$ <u>for</u> $|\lambda| \geq N_0$ <u>and satisfies there</u>

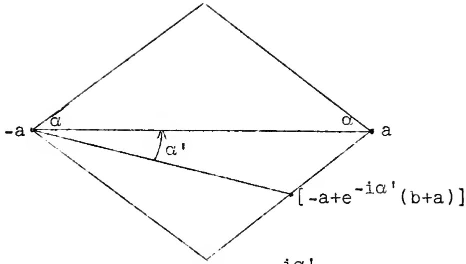
$$|R(\lambda)| = O(|\lambda|^s)$$

<u>for some integer</u> $s \geq -1$. <u>Then if</u> $u(-a)$ <u>belongs to the domain of</u> A^{s+2} <u>the equation</u>

$$(e^{i\alpha'}D_t - A)v = 0 ,$$

<u>with</u> α' <u>a constant</u> $|\alpha'| < \alpha$, <u>has a solution on some interval</u> $-a < t < b$ <u>with</u> $\lim_{t \to -a} v(t) = u(-a)$. <u>If, furthermore,</u> $u(-a)$ <u>belongs to the domain of</u> A^{s+3} <u>then</u> v <u>is also a solution on the interval</u> $-a \leq t < b$.

<u>Proof</u>: The function u(t) given by (12.22) is an analytic extension of our given solution to the rhombus



and hence the function $v(t) = u(-a + e^{-i\alpha'}(t+a))$ is a solution of (12.23) on an interval $-a < t < b$. We wish to verify that as $t \to -a$, t real $> -a$, $u(-a + e^{-i\alpha'}(t+a))$ tends strongly to $u(-a)$ assuming that $u(-a)$ belongs to the domain of A^{s+2}. (The proof of the last statement in the theorem is similar, and we omit it.)

To show this it suffices to show that the function $u(t)$ given by (12.22) is uniformly continuous in the small triangle Δ: $|\arg (t+a)| \leq \alpha'$, $|t+a) < \varepsilon$ for some small ε. This is clearly so for the terms in (12.22) not involving $u(-a)$ so we consider only the remaining terms

$$u_1(t) = -i \int_{N_0}^{N_0+e^{i\alpha}\cdot\infty} e^{i\lambda(t+a)} R(\lambda) u(-a) d\lambda$$

$$- i \int_{-N_0+e^{i(\pi-\alpha)}\cdot\infty}^{-N_0} e^{i\lambda(t+a)} R(\lambda) u(-a) d\lambda \ .$$

We make use of the following simple identity: for any integer $s \geq -1$,

$$R(\lambda) = \frac{1}{\lambda} + \frac{A}{\lambda^2} + \ldots + \frac{A^{s+1}}{\lambda^{s+2}} + \frac{A^{s+2}}{\lambda^{s+2}} R(\lambda) \ .$$

Substituting this into the preceding integrals we find

$$iu_1(t) = u(-a)\int_{\Gamma} \frac{e^{i\lambda(t+a)}}{\lambda}\,d\lambda$$

$$+ \sum_{k=1}^{s+1} \int_{\Gamma} \frac{e^{i\lambda(t+a)}}{\lambda^{k+1}}\,d\lambda\cdot A^k u(-a) + \int_{\Gamma} \frac{R(\lambda)}{\lambda^{s+2}}\,A^{s+2}u(-a)d\lambda$$

where Γ consists of the two straight lines occurring in the integrals in $u_1(t)$, oriented with Re λ increasing. Each of these integrals, with the exception of the first, is absolutely and uniformly convergent for t in the triangle Δ, since $|R(\lambda)| = O(\lambda^s)$, while the first integral

$$u(-a)\int_{\Gamma} e^{i\lambda(t+a)}\,\frac{d\lambda}{\lambda}$$

is easily seen to be equal to

$$-u(-a)\int_{\Gamma_1} \frac{e^{i\lambda(t+a)}}{\lambda}\,d\lambda$$

where Γ_1 is any curve in the upper half plane joining $-N_o$ and N_o. This last expression is clearly uniformly continuous for t in Δ, and the theorem is proved.

Chapter V

Applications to Differential Problems

We shall apply the abstract theory developed previously to differential problems, being mainly interested in properties of solutions of such problems in cylindrical domains. We shall be concerned with a general class of problems which we term weighted elliptic boundary value problems. This class includes both elliptic and many parabolic problems for cylindrical domains as a special case. The applications, starting in §17, will consist in combining results from the abstract theory with various results (in particular, estimates) from the elliptic theory. In §13 we first describe some known results concerning general elliptic boundary value problems and then pass (§§14 and 15) to weighted elliptic boundary value problems in cylindrical domains; these we transform into first order systems in the direction of the generator (§16).

13. Preliminaries

Consider complex valued functions $u(x)$, $x = (x_1,\dots,x_n)$, defined in a domain G in n-dimensional space. The boundary of G is denoted by ∂G and its closure by $\overline{G}$. Set $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$ for $j = 1,\dots,n$, and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$. Here $\alpha = (\alpha_1 + \dots + \alpha_n)$ is a multi-index with integral components $\alpha_i \geq 0$ whose length $\alpha_1 + \dots + \alpha_n$ we denote by $|\alpha|$. A good part of the applications will concern functions defined in cylindrical domains in which case we shall denote the dimension of the space by n+1 and let $(x,t) = (x_1,\dots,x_n,t)$ be the generic point in E_{n+1}.

For functions $u(x)$ in $C^j(G)$ we define the L_p norms ($p \geq 1$):

$$(13.1) \qquad \|u\|_{j,L_p} = \Big(\sum_{|\alpha|\leq j} \int_G |D^\alpha u|^p \, dx\Big)^{1/p} .$$

The completion of $C^j(G)$ under this norm is a Banach space denoted by $H_{j,L_p}(G)$. For $p = 2$ it is also a Hilbert space.

We shall first consider general elliptic boundary value problems. Denote by $\mathcal{A}(x;D)$ ($D = (D_1,\ldots,D_n)$) an elliptic operator of order $2m$ in G:

$$(13.2) \qquad \mathcal{A}(x;D) = \sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha ,$$

where ellipticity, as usual, means that

$$(13.3) \qquad A'(x;\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \neq 0 , \qquad (\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n})$$

for all real vectors $\xi \neq 0$ and all $x \in \overline{G}$. For $n = 2$ we shall always tacitly assume that in addition the following condition holds.

Condition on $\mathcal{A}$: For every pair of linearly independent real vectors ξ, η and $x \in \overline{G}$ the polynomial in s: $\mathcal{A}'(x;\xi + s\eta)$ has exactly m roots with positive imaginary parts.

As is well known this condition is always satisfied if $n \geq 3$ or if $n = 2$ and the leading coefficients are real.

Let there also be given a system of m differential boundary operators $\{B_j\}_{j=1}^m$ of respective order m_j:

$$(13.4) \qquad B_j(x;D) = \sum_{|\alpha|\leq m_j} b_\alpha^j(x) D^\alpha$$

with coefficients defined on the boundary. We shall be interested in solutions u of the boundary value problem:

$$(13.5)\qquad \begin{aligned} \mathcal{A}u &= f \quad \text{in } G\ , \\ B_j u &= 0 \quad \text{on } \partial G\ , \qquad\qquad j = 1,\dots,m\ . \end{aligned}$$

For convenience we shall refer to the triplet of elliptic operator, boundary system and domain as an "elliptic boundary value problem" and shall denote it by $(\mathcal{A},\{B_j\};G)$.

General boundary value problems were considered during the last few years by many authors. The general existence theory depends on a priori estimates for solutions of (13.5). We shall make use of certain estimates derived for different classes of functions by Agmon, Douglis, Nirenberg [1] (see also Browder [1-2]. For the estimates to hold it is necessary that the following algebraic condition be satisfied:

Complementing Condition: At any point x of ∂G let ν denote the normal to ∂G and $\xi \neq 0$ a real vector parallel to the boundary. We require that the polynomials in s: $B_j'(x;\xi + s\nu)$, $j = 1,\dots,m$ (B_j' denoting principal part) be linearly independent modulo the polynomial $\prod_{k=1}^{m} (s - s_k^+(\xi))$ where $s_k^+(\xi)$ are the roots of $\mathcal{A}'(x;\xi + s\nu)$ with positive imaginary parts.

Suppose that the Complementing Condition holds, that the B_j are of order $m_j < 2m$, G bounded, and

Smoothness assumption: G is of class C^{2m}. The leading coefficients of $\mathcal{A}$ are continuous in $\overline{G}$, the other coefficients

being measurable and bounded. The coefficients of B_j $(j = 1,\ldots,m)$ belong to C^{2m-m_j} on the boundary.

Under the above assumptions the following a priori L_p estimates hold.

Theorem 5.1: *Consider the class of functions* u *in* $C^{2m}(\overline{G})$ *satisfying the boundary conditions:*

$$B_j u = 0 \quad \underline{\text{on}} \quad \partial G , \qquad\qquad j = 1,\ldots,m ,$$

and let $1 < p < \infty$. *Then*:

$$\|u\|_{2m,L_p} \leq C\left(\|\mathcal{A}u\|_{L_p} + \|u\|_{L_p}\right) \tag{13.6}$$

where C *is some constant depending on* $(\mathcal{A}, \{B_j\}; G)$ *and* p *but not on* u.

This theorem in a more general form was established in Agmon, Douglis, Nirenberg [1].

One calls a boundary system of differential operators $\{B_j\}$ a *normal* system if

a) The boundary ∂G is non-characteristic to B_j at every point.

b) The orders of the different operators are distinct.

When G is bounded we shall use the following

Definition 5.1: An elliptic boundary value problem $(\mathcal{A}, \{B_j\}_1^m; G)$ is called a *regular* problem if

i) The elliptic operator $\mathcal{A}$ (of order $2m$) and the boundary operator $\{B_j\}$ satisfy the Complementing Condition.

ii) $\{B_j\}$ is a normal boundary system of m operators of orders $\leq 2m-1$.

(iii) The smoothness assumptions on the domain and the coefficients introduced above hold.

We note that Theorem 5.1 holds in particular for regular elliptic boundary value problems.

14. Boundary value problems in cylindrical domains

As mentioned already the applications of the abstract theory will deal mostly with solutions of differential problems in cylindrical domains. The problems we have in mind are general elliptic and parabolic boundary value problems and more generally a class of problems which we term weighted elliptic. Before describing these problems let us modify our notations. We shall denote by n+1 ($n \geq 1$) the space dimension and let $(x,t) = (x_1,\ldots,x_n,t)$ be the generic point in E_{n+1}. We denote by Γ the infinite cylinder: $\Gamma = \{(x,t): x \in G, -\infty < t < \infty\}$ where G is some bounded domain in E_n (interval for $n = 1$). We call G the base of Γ. We put $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $D_x = (D_1,\ldots,D_n)$ and denote by D_x^α ($\alpha = (\alpha_1,\ldots,\alpha_n)$) the x-derivative $D_1^{\alpha_1} \ldots D_n^{\alpha_n}$. We shall be interested in linear differential operators of the form:

$$(14.1)\quad \mathcal{A}(x;D_x,D_t) = \mathcal{A}_\ell(x;D_x) + \mathcal{A}_{\ell-1}(x;D_x)D_t + \ldots + \mathcal{A}_0 D_t^\ell$$

where the $\mathcal{A}_j(x;D_x)$ ($j \geq 1$) are linear differential operators in x with variable coefficients defined in G and $\mathcal{A}_0$ a non-zero

constant. Denote by s_j the order of $\mathcal{A}_j$. We shall say that $\mathcal{A}$ is of order type $(2m,\ell)$ if the following holds:

$$(14.2) \qquad s_\ell = 2m \ , \quad \text{and in general} \quad s_j \leq \frac{2m}{\ell} j \ .$$

Denote by $\mathcal{A}_j^{\#}(x;D_x)$ $(j = 0,\ldots,\ell)$ the sum of terms in $\mathcal{A}_j$ which are of precise order $\frac{2m}{\ell}j$, letting $\mathcal{A}_j^{\#} \equiv 0$ if there are no such terms. Clearly, $\mathcal{A}_0^{\#} = \mathcal{A}_0 =$ constant, $\mathcal{A}_\ell^{\#} = \mathcal{A}_\ell'$ and $\mathcal{A}_j^{\#} \equiv 0$ if $\frac{2mj}{\ell}$ is not an integer. We now define the weighted principal part of $\mathcal{A}^{\#}$ of $\mathcal{A}$ as:

$$(14.3) \qquad \mathcal{A}^{\#}(x;D_x,D_t) = \sum_{j=0}^{\ell} \mathcal{A}_{\ell-j}^{\#}(x;D_x)D_t^j \ .$$

With these notations we introduce the following

<u>Definition 5.1</u>: An operator $\mathcal{A}(x;D_x,D_t)$ which is of order type $(2m,\ell)$ is said to be a weigthed elliptic operator in Γ $(\bar{\Gamma})$ if

$$(14.4) \qquad \mathcal{A}^{\#}(x;\xi,\tau) \neq 0$$

for all real vectors $(\xi,\tau) = (\xi_1,\ldots,\xi_n,\tau) \neq 0$ and all $x \in \Gamma$ $(\bar{\Gamma})$.

A weighted elliptic operator of order type $(2m,2m)$ is simply an elliptic operator in (x,t) of order $2m$. On the other hand weighted elliptic operators of order $(2m,1)$ include the standard parabolic operators. We also observe that if $\mathcal{A}$ is a weighted elliptic operator of order type $(2m,\ell)$, then $\mathcal{A}_\ell(x;D_x)$ is an ordinary elliptic operator in x of order $2m$. The operator $(\frac{\partial}{\partial t} - \Delta_x)(\frac{\partial}{\partial t} + \Delta_x)$, with Δ_x = the Laplace operator in the x variables

is a weighted elliptic operator, as is each of its factors, while the Schrödinger operator $D_t + \Delta_x$ is <u>not</u> weighted elliptic.

If $n = 1$ we shall always assume in the paper that the weighted elliptic operator satisfies the following

<u>Condition on $\mathcal{A}$</u>: For every real $\tau \neq 0$ and $x \in G$, the polynomial in ξ_1: $\mathcal{A}^{\#}(x;\xi_1,\tau)$ has exactly m roots with a positive imaginary part.

Denote by Γ_a^b the part of Γ contained in $a < t < b$. We shall be interested in solutions v of the boundary value problem:

(14.5) $\quad \mathcal{A}(x;D_x,D_t)v(x,t) = f(x,t) \quad$ in Γ_a^b ,

$$B_j(x;D_x,D_t)v = 0 \quad \text{on curved part (i.e. cylinder side) of } \partial\Gamma_a^b , \quad j = 1,\dots,m ,$$

where $\mathcal{A}$ is a weighted elliptic operator of order type $(2m,\ell)$, and $\{B_j(x;D_x,D_t)\}_1^m$ is a given system of differential operators defined on $\partial\Gamma$. As indicated we assume that the coefficients of B_j are independent of t and thus actually defined on ∂G. We shall again refer to the triplet $(\mathcal{A},\{B_j\};\Gamma)$ as a weighted elliptic boundary value problem.

We restrict the class of weighted elliptic boundary value problems by introducing the analogous

<u>Complementing Condition on</u> $(\mathcal{A},\{B_j\}_1^m;\Gamma)$: At any point (x,t) of $\partial\Gamma$ let ν be the normal to $\partial\Gamma$ and $(\xi,\tau) \neq 0$ be a real vector parallel to $\partial\Gamma$ at the point. We require that the polynomials in s: $B_j'(x;\xi+s\nu,\tau)$, $j = 1,\dots,m$, be linearly independent modulo

the polynomial $\prod_1^m (s - s_k^+(\xi,\tau))$ where $s_k^+(\xi,\tau)$, $k = 1,\ldots,m$, are the roots of $\mathcal{A}^\#(x;\xi+s\nu,\tau)$ with positive imaginary parts.

We note that if $\mathcal{A}$ is an ordinary elliptic operator in (x,t) then the Complementing Condition coincides with the corresponding condition given in the previous section. In the general situation the condition makes sense since ν is also the normal to ∂G at x and it is readily seen from our assumptions that $\mathcal{A}^\#(x;\xi+s\nu,\tau)$ is a polynomial of order $2m$ in s with no real roots such that exactly half the roots possess a positive imaginary part.

Finally, in a manner analogous to the definition of regularity for elliptic boundary value problems in bounded domains, we introduce

Definition 5.2: A weighted elliptic boundary value problem $(\mathcal{A},\{B_j\}_1^m;\Gamma)$ of order $(2m,\ell)$ will be called a regular problem if:

(i) The Complementing Condition holds.

(ii) $\{B_j\}$ is a normal boundary system and the order m_j [1] of B_j is $\leq 2m-1$.

Also the following smoothness conditions should hold:

(iii) G is a bounded domain of class C^{2m}.

(iv) The coefficients of $\mathcal{A}^\#$ are continuous in G, the other coefficients of $\mathcal{A}$ being measurable and bounded.

(v) The coefficients of B_j belong to C^{2m-m_j} on ∂G.

---

[1] When $\partial\Gamma$ is not connected m_j could take different integral values on the different connected components of $\partial\Gamma$ as one can take different boundary systems on the various components. This applies in particular to the special case $n = 1$ in which case $\partial\Gamma$ is composed of two parallel lines.

In the following even if not explicitly stated we shall consider only <u>regular</u> weighted elliptic boundary value problems. Moreover, we shall impose another restriction on the boundary system. Namely, B_j contains no t-differentiation so that the system is of the form $\{B_j(x;D_x)\}$. The restriction to <u>regular</u> problems is necessary because of the existence theory to be used later although some theorems which do not use the existence theory but only the a priori estimates remain valid in the more general situation. On the other hand, the restriction on the boundary system is less essential and is made so that in the reduction of the problems to the abstract forms of the previous chapters the domain of definition of the operator A will be independent of t. By modifying our proofs, somewhat, most of the results could be established without the additional restriction on the boundary system.

The following theorem is basic for the applications.

<u>Theorem 5.2</u>: <u>Let</u> $(\mathcal{A}(x;D_x,D_t),\{B_j(x;D_x)\}_1^m;\Gamma)$ <u>be a regular weighted elliptic boundary value problem of order type</u> $(2m,\ell)$ <u>and</u> $1 < p < \infty$. <u>Set</u> $d = \frac{2m}{\ell}$. <u>Then, for all functions</u> $u(x)\in C^{2m}(\overline{G})$ <u>satisfying the boundary conditions:</u>

$$B_j(x;D_x)u = 0 \quad \text{on} \quad \partial G \;, \qquad j = 1,\ldots,m \;,$$

<u>and for all real λ such that</u> $|\lambda| \geq N_0$, <u>the following estimate holds:</u>

$$(14.6)\quad \sum_{j=0}^{2m} |\lambda|^{(2m-j)/d}\|u\|_{j,L_p(G)} \leq C\|\mathcal{A}(x;D_x,\lambda)u\|_{L_p(G)} \;,$$

<u>where</u> C <u>and</u> N_o <u>are constants depending only on</u> $(\mathcal{A}, \{B_j\}; \Gamma)$ <u>and</u> p <u>but not on</u> u <u>or</u> λ.

Before proving the theorem we remark that from the proof it will be seen that the restriction on the boundary system to be normal is not necessary. Also, one can easily extend the theorem to boundary systems of the form $\{B_j(x;D_x,D_t)\}$.

Theorem 5.2 contains Theorem 12.8 of Agmon, Douglis, Nirenberg [1] as a very special case, and the proof here is much simpler. The theorem is deduced by a simple artifice from Theorem 5.1 applied in n+1 dimensions.

<u>Proof</u>: We shall first prove Theorem 5.2 in the special case $\ell = 2m$, i.e. when $\mathcal{A}$ is an elliptic operator in (x,t) of order 2m. Let $\zeta(t)$ be a C^∞ function on the line such that $\zeta \equiv 1$ for $|t| \leq 1$, $\zeta \equiv 0$ for $|t| \geq \frac{3}{2}$. Let u(x) satisfy the conditions of the theorem, and introduce the function:

$$v(x,t) = u(x)e^{i\lambda t}\zeta(t) \tag{14.7}$$

where λ is some real number. Clearly, $v \in C^{2m}(\bar{\Gamma})$, $v \equiv 0$ for $|t| \geq \frac{3}{2}$ and $B_j v = 0$ on $\partial\Gamma$, $j = 1,\ldots,m$. Hence, denoting by Γ_r $(= \Gamma^r_r)$ the part of the cylinder Γ contained in $|t| < r$, it follows readily that Theorem 5.1 is applicable to v in Γ_2, so that

$$\|v\|_{2m,L_p(\Gamma_2)} \leq C_1\left(\|\mathcal{A}v\|_{L_p(\Gamma_2)} + \|v\|_{L_p(\Gamma_2)}\right). \tag{14.8}$$

Now, $\mathcal{A}v(x,t) = \zeta(t)e^{i\lambda t}\mathcal{A}(x;D_x,\lambda)u(x)$ + linear combination of derivatives of $u(x)e^{i\lambda t}$ of order $\leq$ 2m-1 with coefficients which

are some bounded fixed functions times powers of λ. Using this, and noting that $v = ue^{i\lambda t}$ for $|t| \leq 1$, $|e^{i\lambda t}| = 1$, we obtain from (14.8)

(14.8)' $$\|ue^{i\lambda t}\|_{2m,L_p(\Gamma_1)} \leq \|v\|_{2m,L_p(\Gamma_2)}$$
$$\leq C_2\left(\|\mathcal{A}(x;D_x,\lambda)u\|_{L_p(G)} + \sum_{j=0}^{2m-1} \|u\|_{j,L_p(G)}|\lambda|^{2m-1-j}\right)$$

with some constant C_2 independent of u and λ. Now:

$$\|ue^{i\lambda t}\|^p_{2m,L_p(\Gamma_1)} = \int_{\Gamma_1} \sum_{|\alpha|\leq 2m} |D^\alpha(ue^{i\lambda t})|^p \, dxdt$$
$$\geq \sum_{j=0}^{2m} \int_G \sum_{|\alpha|\leq j} |D_x^\alpha u|^p |\lambda|^{p(2m-j)} dx$$
$$= \sum_{j=0}^{2m} \|u\|^p_{j,L_p(G)}|\lambda|^{p(2m-j)} .$$

From the last inequality and (14.8)' we find for suitable constants C_3, C_4:

(14.9) $$\sum_{j=0}^{2m} \|u\|_{j,L_p(G)}|\lambda|^{2m-j} \leq C_3\|v\|_{2m,L_p(\Gamma_2)}$$
$$\leq C_4\|\mathcal{A}(x;D_x,\lambda)u\|_{L_p(G)} + \frac{C_4}{|\lambda|}\sum_{j=0}^{2m-1} \|u\|_{j,L_p(G)}|\lambda|^{2m-j} .$$

Consequently if $|\lambda| \geq 2C_4$ it follows that (14.6) holds with $C = N_o = 2C_4$. This proves the theorem in this case.

Suppose, now, $\ell \neq 2m$. Let $\mathcal{A}^{\#}$ be the weighted principal part of $\mathcal{A}$ so that by (14.8) the corresponding characteristic form is

$$\mathcal{A}^{\#}(x;\xi,\tau) = \sum_{j=0}^{\ell} \mathcal{A}^{\#}_{\ell-j}(x;\xi)\tau^{j} .$$

Write $d = \frac{2m}{\ell}$ in its lowest terms: $d = \frac{a}{b}$ where a, b are relatively prime positive integers, and put $\ell_1 = \frac{\ell}{b}$. Since $\mathcal{A}^{\#}_{j} = 0$ if $\frac{2jm}{\ell}$ is not an integer, we actually have:

$$(14.10) \qquad \mathcal{A}^{\#}(x;\xi,\tau) = \sum_{j=0}^{\ell_1} \mathcal{A}^{\#}_{\ell-bj}(x;\xi)\tau^{bj} .$$

We now define:

$$(14.11) \qquad \begin{cases} \mathcal{L}^{+}(x;\xi,\tau) = \sum_{j=0}^{\ell_1} \mathcal{A}^{\#}_{\ell-bj}(x;\xi)\tau^{aj} \\ \mathcal{L}^{-}(x;\xi,\tau) = \sum_{j=0}^{\ell_1} (-1)^{bj}\, \mathcal{A}^{\#}_{\ell-bj}(x;\xi)\tau^{aj} . \end{cases}$$

It is readily seen that if $\tau \geq 0$, $\tau^{d} \geq 0$, then:

$$(14.11)' \qquad \begin{aligned} \mathcal{L}^{+}(x;\xi,\tau) &= \mathcal{A}^{\#}(x;\xi,\tau^{d}) \\ \mathcal{L}^{-}(x;\xi,\tau) &= \mathcal{A}^{\#}(x;\xi,-\tau^{d}) . \end{aligned}$$

Also, for each fixed x, $\mathcal{L}^{+}$ and $\mathcal{L}^{-}$ are homogeneous polynomials in (ξ,τ) of degree 2m (we recall that $\mathcal{A}^{\#}_{\ell-bj}$ is homogeneous in ξ of degree $d(\ell-bj) = 2m-aj$). Now, from (14.11)' and the weighted ellipticity of $\mathcal{A}$ it follows that $\mathcal{L}^{\pm}(x;\xi,\tau)$ is different from zero for all real $(\xi,\tau) \neq 0$ with $\tau \geq 0$. From the homogeneity of

$\mathcal{L}^{\pm}$ it then follows that $\mathcal{L}^{\pm}(x;\xi,\tau) = \mathcal{L}^{\pm}(x;-\xi,-\tau) \neq 0$ for <u>all</u> real $(\xi,\tau) \neq 0$. In the same way one sees if $n = 1$ that $\mathcal{L}^{\pm}(x;\xi_1,\tau)$ possesses exactly m roots in ξ_1 with positive imaginary parts for every $\tau \neq 0$. Finally, in the general case, one sees in the same manner that $\mathcal{L}^{\pm}(x;\xi,\tau)$ and the $\{B_j(x;\xi)\}$ satisfy the Complementing Condition. Thus, $(\mathcal{L}^{+}(x;D_x,D_t),\{B_j(x;D_x)\};\Gamma)$ and $(\mathcal{L}^{-},\{B_j\};\Gamma)$ are <u>regular</u> elliptic boundary value problems in (x,t) of order 2m. Since these problems correspond to the case $\ell = 2m$ for which the theorem was already established, it follows readily, using (14.11)' for $\lambda > 0$ and $\lambda < 0$, that for all real λ, $|\lambda| \geq N_0$:

$$(14.12)\qquad \sum_{j=0}^{2m} |\lambda|^{\frac{\ell}{2m}(2m-j)} \|u\|_{j,L_p(G)} \leq C\|\mathcal{A}^{\#}(x;D_x,\lambda)u\|_{L_p(G)} .$$

Now

$$(14.13)\qquad \mathcal{A}^{\#}(x;D_x,\lambda)u = \mathcal{A}(x;D_x,\lambda)u + \sum_{j=0}^{\ell} \lambda^j(\mathcal{A}_{\ell-j}(x,D_x) - \mathcal{A}^{\#}_{\ell-j}(x,D_x))u .$$

From the definition of the weighted principal part it follows readily that a k^{th} order derivative of u in the last sum has its λ factors raised to powers $j < \frac{\ell}{2m}(2m-k)$, so that

$$(14.13)'\qquad \|\mathcal{A}^{\#}(x;D_x,\lambda)u\|_{L_p(G)} \leq \|\mathcal{A}(x;D_x,\lambda)u\|_{L_p(G)} + \frac{K}{|\lambda|^{\frac{1}{2m}}} \sum_{j=0}^{2m} |\lambda|^{\frac{\ell}{2m}(2m-j)} \|u\|_{j,L_p(G)} ,$$

where K depends only on the coefficients of $\mathcal{A} - \mathcal{A}^{\#}$. Clearly it follows from (14.13)' that the inequality (14.12) holds with $\mathcal{A}^{\#}$

replaced by $\mathcal{A}$ if one replaces N_o and C by some larger constants. This completes the proof of the theorem.

We mention that Theorem 5.2 admits a kind of converse. Namely, we have

Theorem 5.2': Let $(\mathcal{A}(x;D_x,D_t),\{B_j(x,D_x)\}_1^m;\Gamma)$ be a weighted elliptic boundary value problem of order type $(2m,\ell)$ such that $\{B_j\}$ is a normal system of boundary operators of orders $\leq 2m-1$ and the usual smoothness assumptions hold. Suppose that for $p = 2$ the conclusion of Theorem 5.2 holds. Then, $(\mathcal{A},\{B_j\}_1^m;\Gamma)$ is a regular problem (i.e. the Complementing Condition holds).

We shall give a brief indication of the proof under the additional assumption that the highest power of 2 which divides ℓ also divides 2m (this will apply in particular when ℓ divides 2m). In this case one can without loss of generality assume further that $\ell = 2m$. (This is done by considering the problem $(\mathcal{L}^+,\{B_j\}_1^m;\Gamma)$ where $\mathcal{L}^+$ is defined by (14.11)' noting that for this problem $\ell = 2m$ and it satisfies the conditions of the theorem since b is odd.) Let now $v(x,t) \in C^{2m}(\bar{\Gamma})$ and assume that:

(i) $v(x,t)$ is periodic in t of period 2π.

(ii) $B_j v = 0$ on $\partial\Gamma$, $j = 1,\ldots,m$.

Consider the Fourier expansion in t:

$$v(x,t) \sim \sum_{-\infty}^{\infty} u_n(x)e^{int}$$

so that

$$\mathcal{A}(x;D_x,D_t)v \sim \sum_{-\infty}^{\infty} \mathcal{A}(x;D_x,n)u_n(x)e^{int} \,. \tag{14.14}$$

From (14.14), Parseval 's formula, and the assumed a priori estimate (14.6) which the u_n satisfy ($p = 2$, $\ell = 2m$), one obtains readily that the class of functions v satisfy the following estimate:

$$(14.15)\quad \|v\|_{2m,L_2(\Gamma_\pi)} \leq \text{const}\left(\|\mathcal{A}(x;D_x,D_t)v\|_{L_2(\Gamma_\pi)} + \|v\|_{L_2(\Gamma_\pi)}\right)$$

This estimate, however, implies that the Complementing Condition holds. This is shown by simple counterexamples, and the proof is just a refinement of the proof of the necessity of the Complementing Condition in a very similar situation given in Agmon, Douglis, Nirenberg [1].

For $p = 2$ Theorem 5.2 yields an estimate for regular weighted elliptic boundary value problems.

<u>Corollary</u>: Let $(\mathcal{A},\{B_j\}_1^m;\Gamma)$ be as in Theorem 5.2 and let $u(x,t) \in C^{2m}(\Gamma_a^b)$ be a function satisfying the boundary conditions $B_j u = 0$, $j = 1,\ldots,m$ on the cylinder side. If $a < a' < b' < b$ the inequality

$$\sum_{\substack{j,k\geq 0 \\ \frac{j}{\ell}+\frac{k}{2m}\leq 1}} \int_{a'}^{b'}\int_G |D_t^j D_x^k u|^2 dxdt$$
$$\leq \text{constant}\int_a^b\int_G |Au|^2 dxdt + \text{constant}\int_a^b\int_G |u|^2 dxdt$$

holds with some constant independent of u.

<u>Proof</u>: Assume first that u and its derivatives up to order 2m vanish at $t = a$ and $t = b$. Then on taking Fourier transforms with respect to t the result follows immediately (in fact with $a' = a$, $b' = b$) from (14.6) with the aid of Parseval's theorem.

To treat the general case let $\zeta(t)$ be a C^∞ function of t which equals one in the interval (a',b') and vanishes outside the interval (a,b), and consider the function $v = \zeta^{2m}u$. By applying the result just obtained to the function v one obtains the desired result with the aid of some elementary inequalities.

The proper, more complete, formulation of this result involves fractional derivatives with respect to t and may be extended to operators with coefficients depending on t by means of partition of unity, and using the methods of Agmon, Douglis, Nirenberg [1]. For general results see Peetre [1].

The corollary has the consequence that the completion in the L_2 norm over Γ^+ of solutions of $\mathcal{A}u = 0$ in Γ^+, satisfying $B_j u = 0$ on the side of the cylinder is an interior compact space in the sense of Lax [3].

15. Application to the existence theory

We have introduced in §13 the Banach space $H_{k,L_p}(G)$ consisting of functions $u \in L_p(G)$ possessing generalized (strong) derivatives in $L_p(G)$ up to the order k. When one considers boundary value problems it is natural to consider subspaces of H_{k,L_p} consisting of functions satisfying linear boundary conditions in some generalized sense. Thus in particular if $\{B_j\}$ is a system of linear differential operators of orders < k defined on ∂G we shall denote by $H_{k,L_p}(G;\{B_j\})$ the completion in the norm of H_{k,L_p} of all functions $u \in C^k(\overline{G})$ which satisfy the boundary conditions:

$$B_j u = 0 \quad \text{on} \quad \partial G \quad \text{for all} \quad j .$$

Remark: It is obvious that under the respective conditions of Theorems 5.1, 5.2 the basic estimates (13.6) and (14.6) hold for all functions $u(x)$ in $H_{2m,L_p}(G;\{B_j\}_1^m)$.

Let $(\mathcal{A}(x;D),\{B_j(x;D)\}_1^m;G)$ be an elliptic boundary value problem of order 2m. The L_p existence theory is concerned with the problem: Given $f \in L_p(G)$ find $u \in H_{2m,L_p}(G;\{B_j\})$ such that $\mathcal{A}u = f$. More generally, determine the class of functions f which admit a solution u. The existence theory was developed recently by a number of authors for regular[1] elliptic boundary value problems (Schechter [1] for $p = 2$; Browder [1-2] and Agmon[2] for general $p > 1$). If one considers $\mathcal{A}$ as a closed operator in $L_p(G)$ with domain $H_{2m,L_p}(G;\{B_j\})$ one can show that: (i) the null space of $\mathcal{A}$ is finite dimensional (this is an immediate consequence of (13.6)). (ii) The range of $\mathcal{A}$ is closed and its co-dimension is finite. Furthermore, under additional smoothness assumptions one shows that the formally adjoint problem is also a regular problem and that the "alternative" holds for the two problems. It is not established, however, by the above theories that for an arbitrary regular problem the spectrum of $\mathcal{A}$ is not the whole complex plane, nor that the dimension of the null space of $\mathcal{A}$ and the co-dimension of its range are equal (a result which is probably false in general).

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[1] Existence results for a wider class of problems were obtained by Schechter [2] and more completely by J. Peetre [1].

[2] To be published. The method is based on a regularity theory in L_p which for the Dirichlet problem was described in Agmon [1].

We shall display a subclass of regular problems possessing this property. Moreover, a problem in this subclass can be "imbedded" in a family of regular problems $(\mathcal{A}^\lambda, \{B_j\}; G)$ $(\mathcal{A}^0 = \mathcal{A})$ depending in a polynomial way on a complex parameter λ such that for all values of the parameter with the exception of a discrete set the mapping $u \to \mathcal{A}^\lambda u$ is one-to-one from $H_{2m,L_p}(G;\{B_j\})$ onto $L_p(G)$. Our subclass consists of restrictions to the x variable of regular weighted elliptic problems in one more variable in a cylinder Γ erected over G.

Let $(\mathcal{A}(x;D_x,D_t), \{B_j(x;D_x)\}_1^m; \Gamma)$ be a regular weighted elliptic boundary value problem of order type $(2m,\ell)$ defined in a cylindrical domain Γ with base G. Write $\mathcal{A}$ in the form:

$$\mathcal{A}(x;D_x,D_t) = \mathcal{A}_\ell(x;D_x) + \mathcal{A}_{\ell-1}(x;D_x)D_t + \ldots + \mathcal{A}_0 D_t^\ell .$$

Then $\mathcal{A}_\ell$ is an elliptic operator in x in the domain G and $(\mathcal{A}_\ell, \{B_j\}_1^m; G)$ is a regular elliptic boundary value problem of order 2m in G. We shall say that $(\mathcal{A}_\ell, \{B_j\}; G)$ is the x-restriction of $(\mathcal{A}, \{B_j\}; \Gamma)$. More generally, for a complex λ set:

$$\mathcal{A}^\lambda(x;D_x) = \mathcal{A}(x;D_x,\lambda) \tag{15.1}$$

(so that $\mathcal{A}^0 = \mathcal{A}_\ell$). We shall refer to the family of problems $(\mathcal{A}^\lambda, \{B_j\}; G)$ depending on the complex parameter λ as the <u>reduced</u> weighted elliptic problem. All these problems are regular and as a matter of fact possess the same principal part. For this family of problems the existence theory takes a particularly simple form. The basic result here is

<u>Theorem 5.3</u>: <u>Let</u> $(\mathcal{A}^{\lambda}(x;D),\{B_j(x;D)\};G)$ <u>be a reduced (regular) weighted elliptic problem of order type</u> $(2m,\ell)$ <u>and let</u> $1 < p < \infty$. <u>Then the mapping</u>: $u \longrightarrow \mathcal{A}^{\lambda}u$ <u>is one-to-one from</u> $H_{2m,L_p}(G;\{B_j\})$ <u>onto</u> $L_p(G)$ <u>for all real</u> λ <u>sufficiently large in absolute value</u>.

The part of the theorem which asserts that the mapping is one-to-one is a direct consequence of Theorem 5.2 which, as was already remarked, applies to all $u \in H_{2m,L_p}(G;\{B_j\})$. The second part which asserts that the mapping is onto will be given in the paper of Agmon (to appear) on existence theory. The proof consists in showing directly for real λ sufficiently large that if $u'(x) \in L_{p'}(G)$ $(\frac{1}{p'} + \frac{1}{p} = 1)$ is orthogonal to all functions $\mathcal{A}^{\lambda}u$, $u \in H_{2m,L_p}(G;\ B_j\ )$, then u' is a null function. The theorem could also be derived in a less straightforward manner and under additional smoothness assumptions from the results of Schechter and Browder[1] referred to above.

<u>Remark</u>: Suppose that $(\mathcal{L},\{B_j\};G)$ is a regular elliptic boundary value problem of order 2m which is the x-restriction of some weighted elliptic problem $(\mathcal{A},\{B_j\};\Gamma)$ in the (x,t) space. It follows from Theorem 5.3 that one can add to $\mathcal{L}$ a lower order

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[1] If the coefficients of $(\mathcal{A},\{B_j\};\Gamma)$ are sufficiently smooth then the formally adjoint problem $(\mathcal{A}^*,\{B_j^*\};\Gamma)$ exists and can be shown also to be a regular problem. One notes further that the formal adjoint of the reduced problem is the reduced problem of $(\mathcal{A}^*,\{B_j^*\};\Gamma)$. Consequently by Theorem 5.2, the uniqueness part of Theorem 5.3 holds for both $(\mathcal{A}^{\lambda},\{B_j\};G)$ and for its formal adjoint. The complete result follows now by applying the "alternative" of the existence theory.

operator obtaining an operator $\tilde{\mathcal{L}}$ such that the mapping $u \to \tilde{\mathcal{L}}u$ is one-to-one from $H_{2m,L_p}(G;\{B_j\})$ onto $L_p(G)$ (one takes $\tilde{\mathcal{L}} = \mathcal{A}^\lambda$ with real λ sufficiently large). From this it follows easily, by a standard reduction to the Fredholm alternative for compact operators, that the dimension of the null space of $\mathcal{L}$ (with domain $H_{2m,L_p}(G,\{B_j\})$ in L_p) and the co-dimension of its range are equal).

16. Reduction of differential problems in cylindrical domains to the abstract form. Properties of the resolvent. Regularity

Let $(\mathcal{A}, \{B_j\}_1^m; \Gamma)$ be a regular weighted elliptic differential boundary value problem of order type $(2m,\ell)$. Dividing if necessary by a constant we assume from now on that $\mathcal{A}$ has the form:

$$(16.1)\qquad \mathcal{A}(x;D_x,D_t) = D_t^\ell + \mathcal{A}_1(x;D_x)D_t^{\ell-1} + \dots + \mathcal{A}_\ell(x;D_x) .$$

We shall be interested in solutions $u(x,t)$ in cylindrical sections Γ_a^b (the part of Γ in $a < t < b$) which satisfy:

$$(16.2)\qquad \begin{aligned} &\mathcal{A}(x;D_x,D_t)u(x,t) = f(x,t) \quad \text{in } \Gamma_a^b \\ &B_j(x;D_x)u = 0 \quad \text{on curved part of } \partial\Gamma_a^b , \quad j = 1,\dots,m . \end{aligned}$$

By introducing as new unkowns the functions $u_j = D_t^j u$, $j = 0,\dots,\ell-1$, one can write the first equation (16.2) as a system (as in §2)

$$(16.2)'\qquad \begin{aligned} &D_t u_j - u_{j+1} = 0 , \qquad\qquad j = 0,\dots,\ell-2 , \\ &D_t u_{\ell-1} + \sum_{j=0}^{\ell-1} \mathcal{A}_{\ell-j} u_j = f . \end{aligned}$$

If we introduce the vectors $U = (u_o, u_1, \ldots, u_{\ell-1})$, $F = (f_o, \ldots, f_{\ell-1})$, The system $(16.2)'$ can be written in the more condensed form:

$$(16.3) \qquad D_t U - AU = F$$

where

$$(16.3)' \qquad AU = (u_1, u_2, \ldots, u_{\ell-1}, -\sum_{j=0}^{\ell-1} \mathcal{A}_{\ell-j} u_j)$$

and F is the special vector $(0,0,\ldots,0,f)$. For a general F equation (16.3) is the same as the system:

$$(16.3)'' \qquad \begin{aligned} u_j &= D_t^j u_o - \sum_{k=1}^{j} D_t^{j-k} f_{k-1}, \qquad j = 0,\ldots,\ell-1, \\ Au_o &= f_{\ell-1} + \sum_{k=1}^{\ell-1} \left[D_t^{\ell-k} + \sum_{j=k}^{\ell-1} D_t^{j-k} \mathcal{A}_{\ell-j} \right] f_{k-1}. \end{aligned}$$

The reduction of the higher order problem consists in replacing (16.2) by the first order equation (16.3) where U is a function of t with values in some Banach space. Setting $d = \frac{2m}{\ell}$ we shall usually assume that the values of U lie in the Banach space $\mathcal{B}_p'$ $(p > 1)$ defined as the Cartesian product (ℓ factors):

$$(16.4) \qquad X = \mathcal{B}_p' = H_{2m,L_p}(G) \times H_{2m-d,L_p}(G) \times \ldots \times H_{d,L_p}(G)$$

or else in the space

$$Y = \mathcal{B}_p = H_{2m-d,L_p}(G) \times H_{2m-2d,L_p}(G) \times \ldots \times H_{o,L_p}(G)$$

and that $(D_t - A)U$ lies in the space $\mathcal{B}_p = Y$. Now, in general d need not be an integer in which case (16.4) involves spaces H_{r,L_p}

with non-integral r. The definition of such spaces for p = 2 presents no difficulty (using for instance Fourier transform). A suitable definition for general p is also possible although more involved. Using these H-spaces of functions with fractional derivatives it is possible to define the Banach spaces $\mathcal{B}_p$ in all cases. Nevertheless, for the sake of simplicity we prefer not to deal with these spaces, and for this reason we shall impose from now on the additional

<u>Condition</u>: $\quad d = \frac{2m}{\ell}$ is an integer .

We remark that this condition covers the cases which seem to be the most interesting in applications: $\ell = 1,2$ and $2m$. Note that $\mathcal{B}_p' \subset \mathcal{B}_p$ and in fact that the unit sphere in $\mathcal{B}_p'$ has compact closure in $\mathcal{B}_p$.

The linear operator A introduced above is defined now more precisely as follows: it is a linear (closed) operator with domain:

$$(16.5) \quad \mathcal{D}_A = H_{2m,L_p}(G;\{B_j\}) \times H_{2m-d,L_p}(G) \times \dots \times H_{d,L_p}(G) \; ,$$

in $\mathcal{B}_p'$ such that, for $U \in \mathcal{D}_A$, AU (in $\mathcal{B}_p$) is given by (16.3)' (it is well defined since $\mathcal{A}_{\ell-j}$ is of order $\leq d(\ell-j) = 2m-jd$). By a solution of (16.3) we shall usually mean a function U(t) in an interval (a,b) with values in $\mathcal{D}_A$ such that U(t) is continuous and possesses strongly continuous derivatives (in $\mathcal{B}_p$) and such that (16.3) holds. It will become clear, however, in the following that our results hold in a more general situation when the derivative, for instance, is taken in a generalized sense, D_tU is locally integrable and (16.3) holds almost everywhere.

Returning to the higher order problem (16.2) we shall say that a function $u(x,t)$ in Γ_a^b is a solution if for every fixed t in (a,b), u belongs to $H_{2m,L_p}(G;\{B_j\})$ and if there is a vector $U = (u_o,u_1,\ldots,u_{\ell-1}) \in \mathcal{D}_A$ with $u_o = u$ such that (16.3) holds in the above sense with $F = (0,0,\ldots,0,f)$ ($f \in L_p(G)$ for every fixed t in the interval). We shall refer to u_j as the (generalized) D_t^j derivatives of u.

For a complex λ denote by $R(\lambda;A)$ the resolvent of A when it exists: $R(\lambda;A) = (\lambda I-A)^{-1}$. In the abstract theory properties of solutions of (16.3) were derived under certain assumptions on the resolvent. We shall show that in the concrete case before us the resolvent indeed has many of the stipulated properties.

In the following $|R|_{\mathcal{B}_p'}$ ($|R|_{\mathcal{B}_p}$) denotes the norms of the resolvent R as a linear mapping from $\mathcal{B}_p$ into $\mathcal{B}_p'$ ($\mathcal{B}_p$).

<u>Theorem 5.4</u>: <u>The resolvent</u> $R(\lambda;A)$ <u>as a mapping of</u> $\mathcal{B}_p$ <u>into</u> $\mathcal{B}_p'$ ($\mathcal{B}_p$) <u>exists and is a bounded (compact) operator for all complex</u> λ <u>except for a discrete set</u> $\{\lambda_n\}$ - <u>the eigenvalues of</u> A. <u>As a function of</u> λ, $R(\lambda;A)$ <u>is a meromorphic function with poles at the points</u> λ_n. <u>Furthermore, there exist numbers</u> $0 < \delta < \frac{\pi}{2}$ <u>and</u> $N > 0$ <u>such that the double sector</u> Σ: $|\arg(\pm\lambda)| \le \delta$, $|\lambda| \ge N$, <u>is free of poles and such that in addition the following inequalities hold:</u>

(i) $|R(\lambda;A)|_{\mathcal{B}_p'} + |\lambda|\,|R(\lambda;A)|_{\mathcal{B}_p} \le \text{constant } |\lambda|^{\ell-1}$ <u>in</u> Σ.

(ii) <u>Denote by</u> S <u>the subspace of elements in</u> $\mathcal{B}_p$ <u>of the form</u> $(0,0,\ldots,0,f)$, <u>then:</u>

$$|R_S(\lambda;A)|_{\mathcal{B}_p'} + |\lambda||R_S(\lambda;A)|_{\mathcal{B}_p} \leq \text{constant} \quad \underline{\text{in}} \quad \Sigma ,$$

<u>where</u> R_S <u>is the restriction of</u> $R(\lambda;A)$ <u>to</u> S.

(iii) $$\left|\frac{d}{d\lambda} R_S(\lambda;A)\right|_{\mathcal{B}_p'} + \left|\lambda \frac{d}{d\lambda} R_S(\lambda;A)\right|_{\mathcal{B}_p} = O\left(\frac{1}{\lambda}\right)$$

<u>as</u> $|\lambda| \longrightarrow \infty$ <u>on the real axis</u>.

<u>Proof</u>: The existence of the resolvent means that for every $F \in \mathcal{B}_p$ there exists a unique $U \in \mathcal{D}_A$ such that

(16.7) $$\lambda U - AU = F .$$

Breaking into components: $U = (u_o,\ldots,u_{\ell-1})$, $F = (f_o,\ldots,f_{\ell-1})$, (16.7) can be written as the system:

(16.7)' $$\mathcal{A}(x;D_x,\lambda)u_o = f_{\ell-1} + \sum_{k=1}^{\ell-1}\left(\lambda^{\ell-k} + \sum_{j=k}^{\ell-1}\lambda^{j-k}\mathcal{A}_{\ell-j}\right)f_{k-1} ,$$
$$u_j = \lambda^j u_o - \sum_{k=1}^{j}\lambda^{j-k}f_{k-1} , \qquad j = 1,\ldots,\ell-1 .$$

Clearly (16.7) will have a unique solution if and only if the first equation (16.7)' admits a unique solution $u_o \in H_{2m,L_p}(G;\{B_j\})$. However, by Theorem 5.3 this is precisely the case for real λ of sufficiently large modulus. Thus, for such λ, $R(\lambda;A)$ exists and is a bounded (compact) mapping of $\mathcal{B}_p$ into $\mathcal{B}_p'$ ($\mathcal{B}_p$). Now, this implies that $R(\lambda;A)$ exists for all λ except for a discrete sequence $\{\lambda_n\}$ which are the eigenvalues of A. Moreover, $R(\lambda;A)$ is a meromorphic function of λ with poles λ_n. For if $T = R(\lambda_o;A)$ for some λ_o in the resolvent set of A we have

$$R(\lambda;A) = \frac{1}{\lambda_o - \lambda} TR(\frac{1}{\lambda_o - \lambda};T)$$

and these results are well known for the resolvent of the compact operator T.

Introduce now the family of differential problems $W_\theta = (\mathcal{A}(x;D_x,e^{i\theta}D_t),\{B_j(x;D_x)\}_1^m;\Gamma)$ depending on the real parameter θ. Since W_θ is the given regular weighted elliptic boundary value problem it follows (by continuity) that if $\delta > 0$ is sufficiently small then all the problems W_θ with $|\theta| \leq \delta$ are regular weighted elliptic boundary value problems of order type $(2m,\ell)$. Moreover, it is readily seen that for δ sufficiently small the estimate (14.6) of Theorem 5.2 holds uniformly for the family W_θ, $|\theta| \leq \delta$. In other words, it follows that there exist constants $0 < \delta < \frac{\pi}{2}$ and $N > 0$ such that in the double sector $|\arg(\pm\lambda)| \leq \delta$, $|\lambda| \geq N$, the following inequality holds for all $u \in H_{2m,L_p}(G;\{B_j\})$:

$$(16.8) \qquad \sum_{j=0}^{2m} |\lambda|^{\frac{2m-j}{d}} \|u\|_{j,L_p} \leq \text{constant} \, \|\mathcal{A}(x;D_x,\lambda)u\|_{L_p}$$

with a constant not depending on u or λ. From (16.8) and (16.7)' it follows that there are no eigenvalues of A in Σ (so that by the above $R(\lambda;A)$ exists everywhere in Σ). Moreover, using the same relations one checks easily that the estimates (i) and (ii) of the theorem hold. This completes the proof of (i) and (ii).

The assertion (iii) follows directly from (ii) with the aid of the Cauchy integral theorem.

<u>Definition. Lower Order</u>: Let T be an unbounded linear operator in a Banach space such that $R(\lambda;T)$ is a meromorphic function in λ.

The lower order of the resolvent is defined as the smallest non-negative number $\omega = \omega(T)$ with the following property: Given $\varepsilon > 0$ there exists a sequence of circles C_n ($n = 1,2,\ldots,$) in the complex plane, containing the origin and whose distance from the origin tends to infinity, such that $R(\lambda;T)$ exists on C_n and

$$\|R(\lambda;T)\| \leq e^{|\lambda|^{\omega+\varepsilon}} \quad \text{for} \quad \lambda \in C_n \,, \quad n = 1,2,\ldots \,.$$

Suppose on the other hand that T is a compact operator so that $R(\lambda;T)$ is a meromorphic function of $\frac{1}{\lambda}$. In this case we define the lower order $\omega(T)$ as the smallest non-negative number ω with the property: Given $\varepsilon > 0$ there exists a sequence of circles C_n in the λ plane, containing the origin and whose distance from the origin tends to zero, such that $R(\lambda;T)$ exists on C_n and

$$\|R(\lambda;T)\| \leq e^{|\lambda|^{-\omega-\varepsilon}} \quad \text{for} \quad \lambda \in C_n \,, \quad n = 1,2,\ldots \,.$$

We now complement Theorem 5.4 with

<u>Theorem 5.4'</u>: <u>Let</u> A <u>be the operator of Theorem</u> 5.4 <u>and assume that</u> $p = 2$ ($\mathcal{B}_2$ <u>is thus a Hilbert space</u>). <u>Then</u>:

$$\omega(A) \leq \frac{n}{d} \,. \tag{16.9}$$

Theorem 5.4' follows from the results of Agmon [2]. Indeed, let λ_0 be in the resolvent set of A and set $T = R(\lambda_0;A)$. Then T is a compact operator in $\mathcal{B}_2$ which takes $\mathcal{B}_2$ boundedly into $\mathcal{B}_2'$. T satisfies all the conditions of Theorem A.1.1 of Agmon [2], from which it follows that

$$(16.10) \qquad \omega(T) \le \frac{n}{d} .$$

Since the resolvents of A and T are connected by the relation

$$(16.11) \qquad R(\lambda;A) = \frac{1}{\lambda_o - \lambda} R(\frac{1}{\lambda_o - \lambda};T)T .$$

The bound (16.9) follows easily from (16.10).

With the aid of Theorems 5.4, 5.4' we are in a position to apply the abstract theory. The results of Chapter 4 yield immediately the following:

Regularity Theorem 5.5: *Let* u *be a solution of* (16.2) *for* $0 < t < T$ *and suppose that for every* t, $u \in H_{2m,L_p}(G;\{B_j\})$ *and* $f \in H_{o,L_p}(G)$ *and that* f *depends* C^∞ *(analytically) on* t *as an element in* $H_{o,L_p}(G)$. *Then* $u(t,x)$, *regarded as an element of* $H_{2m-d,L_p}(G)$ *is a* C^∞ *(analytic) function of* t *in the interval* $0 < t < T$.

It may be shown that u *is actually* C^∞ *(analytic) as an element of* $H_{2m,L_p}(G;\{B_j\})$. (In fact if one examines the proofs of Theorems 4.3, 4.4 one sees that in the representations (12.15) and (12.21)' for $u(t)$ all terms belong to our space $X = \mathcal{B}_p'$ except possibly for the functions $u_o(t)$. Since, however, our resolvent operator has at most a finite number of poles on the real axis one may again deform the contours occurring in the theorems slightly off the real axis and obtain functions $u_o(t)$ belonging to X.)

We see furthermore from the results of Chapter 4 that if u is a solution of the inhomogeneous Schrödinger equation in a finite cylinder $0 < t < T$

$$\frac{1}{i}\frac{du}{dt} + \Delta_x u = f \ ,$$

and vanishes on the sides of the cylinder and if f is C^∞ or analytic in t the solution u need <u>not</u> be, for the operator A has infinitely many real eigenvalues and so the necessary conditions of Theorems 4.1', 4.2 are not satisfied.

<u>Remark</u>: If we apply Theorem 4.6, with $s = -1$, we see also that if u is a solution of (16.2), with $f = 0$, for $0 \le t < b$, and if the Cauchy data of u at $t = 0$ is sufficiently regular then, for real α' sufficiently small in absolute value there exists a solution v of the problem

$$(x;D_x,e^{i\alpha'}D_t)v(x,t) = 0 \quad \text{for} \quad 0 < t < b'$$

$$B_j(x,D_x)v = 0 \quad \text{on side of cylinder} \ , \quad j = 1,\dots,m$$

in some interval $0 < t < b'$ with the same Cauchy data as u at $t = 0$. To be precise if, for $j \le \ell$, $D_t^j u$ is strongly continuous in H_{2m-jd,L_p} on $o \le t < b$, then there is a solution v of the problem above such that for $j < \ell$, $D_t^j v(t,x)$ tends to $D_t^j u(0,x)$ in the $H_{2m-(j+1)d,L_p}$ topology as $t \longrightarrow 0$.

It may be shown that the convergence holds also in the H_{2m-jd,L_p} topology.

Using the last statement of Theorem 4.6 it may also be shown, assuming the Cauchy data of u at $t = 0$ to be still more regular, that the function v is a solution for $0 \le t < b'$.

We conclude this section with some remarks.

Let $\{\lambda_n\}$ be the sequence of eigenvalues of A. Then, by the above $R(\lambda;A)$ exists for $\lambda \neq \lambda_n$. Taking in particular $F = (0,\ldots,0,f)$ and comparing with (16.7)' we conclude that the equation:

$$\mathcal{A}^\lambda u = \mathcal{A}(x;D_x,\lambda)u = f \tag{16.12}$$

admits a unique solution $u \in H_{2m,L_p}(G;\{B_j\})$ for every $f \in L_p(G)$. Thus we can add a

<u>Complement to Theorem 5.3</u>: The conclusion of Theorem 5.3 holds for all complex λ except for a discrete set $\{\lambda_n\}$.

Let us write the solution of (16.12) in the form

$$u = \tilde{R}(\lambda)f \ , \tag{16.12'}$$

where for each $\lambda \neq \lambda_n$, $\tilde{R}(\lambda) = (\mathcal{A}^\lambda)^{-1}$ is a bounded operator in $L_p(G)$ (mapping $L_p(G)$ into $H_{2m,L_p}(G;\{B_j\})$). We shall refer to $\tilde{R}(\lambda)$ as a generalized resolvent. Due to the relations (16.7)' there is a close connection between $R(\lambda;A)$ and $\tilde{R}(\lambda)$. In particular, using these relations and Theorem 5.4 it follows easily that $\tilde{R}(\lambda)$ is a meromorphic operator valued function in the complex plane with poles at the points λ_n. We note that in a general abstract situation generalized resolvents were first considered by Keldys [1] who also gave applications to non-self-adjoint differential problems. If $R_S(\lambda)$ is the restriction of $R(\lambda;A)$ to the subspace S consisting of vectors f of the form $(0,\ldots,f_{\ell-1})$ then indeed we see that if $U = R_S(\lambda)$, $f \in S$ then

$$u_j = \lambda^j \tilde{R}(\lambda) f_{\ell-1} \ .$$

Thus the poles of $\tilde{R}$ and $R_S(\lambda)$, and their orders, are the same.

Let $\lambda = \lambda_n$ be an eigenvalue of A and $\Phi \neq 0$ the corresponding eigenelement: $\lambda\Phi - A\Phi = 0$, it follows from (16.7)' that Φ is of the form:

$$\Phi = (\phi, \lambda\phi, \ldots, \lambda^{\ell-1}\phi) \tag{16.13}$$

where $\phi \in H_{2m,L_p}(G;\{B_j\})$, $\phi \neq 0$, and

$$\mathcal{A}^\lambda \phi = 0 \ . \tag{16.13$'$}$$

Conversely, if for some λ the equation (16.13)' admits a non-zero solution in $H_{2m,L_p}(G;\{B_j\})$ then λ is an eigenvalue of A with eigenelement given by (16.13). We shall refer to (16.13)' as a higher order eigenvalue problem. We shall still refer to λ for which a non-trivial solution ϕ of (16.13)' exists ($\phi \in H_{2m,L_p}(G;\{B_j\})$) as an eigenvalue and to ϕ as an eigenelement. By the above the sequence of eigenvalues of A coincides with the sequence of eigenvalues of the higher order problem (16.13)'.

17. Asymptotic behavior of solutions

We now apply the abstract theory developed earlier to regular weighted elliptic boundary value problems $(\mathcal{A}, \{B_j\}_1^m; \Gamma)$. Denote by Γ^+ the part of Γ situated in $t > 0$. Our object in this section is to investigate the asymptotic behavior as $t \longrightarrow +\infty$ of solutions $u(x,t)$ of the homogeneous boundary value problem:

$$(17.1)\qquad \begin{cases} \mathcal{A}(x;D_x,D_t)u = 0 & \text{in } \Gamma^+ \\ B_j(x;D_x)u = 0 & \text{on side of } \partial\Gamma^+ , \qquad j = 1,\dots,m . \end{cases}$$

Using the reduction to the abstract first order problem given in the previous section we are led to consider the asymptotic behavior of functions $U(t)$ with values in the Banach space $X = \mathcal{B}'_p$ which are solutions of the equations:

$$(17.1)' \qquad D_tU - AU = 0 \quad \text{for} \quad t > 0 .$$

As was mentioned before, solutions of $(17.1)'$ will be assumed to be strongly continuous and, as elements of $Y = \mathcal{B}_p$, strongly differentiable, the derivative being strongly continuous in Y. This determines the class and the sense of solutions of the higher order equation (17.1) which for convenience we formulate as

<u>Definition</u>: A function $u(x,t)$ will be called a solution of (17.1) if it satisfies the following conditions on the half-line $t > 0$:[1]

(i) u is a function of t with values in $H_{2m,L_p}(G:\{B_j\})$ (we shall also write $u = u(\cdot,t)$). As a function with values in $H_{2m-d,L_p}(G)$, $u(\cdot,t)$ is strongly continuous and possesses (strongly) a strongly continuous derivative D_tu.

(ii) $u(\cdot,t)$ possesses also higher order derivatives in t of orders $j = 2,\dots,\ell$ in the following sense: Let $u_1 = D_tu$. It is assumed that $u_1(\cdot,t)$, as a function with values in $H_{2m-2d,L_p}(G)$, is

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[1] If one considers solutions in a cylindrical section Γ_a^b the conditions should hold for $a < t < b$.

strongly differentiable, the derivative being strongly continuous. Continuing in this manner step by step having defined $u_j = D_t^j u$ as a strongly continuous function with value in $H_{2m-jd,L_p}(G)$ we assume (for $j < \ell$) that as a function with values in $H_{2m-(j+1)d,L_p}(G)$, $u_j(\cdot,t)$ possesses (strongly) a strongly continuous derivative and we let $u_{j+1} = D_t u_j = D_t^{j+1}u$.

(iii) For each fixed t the following relation holds:

$$u_\ell + \sum_{j=0}^{\ell-1} \mathcal{A}_{\ell-j} u_j = 0 \quad (\text{in } L_p(G))$$

where we assume that $\mathcal{A}$ is given by (16.1) and the $\mathcal{A}_j$ are the corresponding differential operators in x.

<u>Remark</u>: We shall later show that actually any solution of (17.1) when considered as a function of t with values in the Banach space $H_{2m,L_p}(G;\{B_j\})$ is analytic in t for t > 0. This is even true for solutions which a priori are taken in some weaker sense.

With the above definition of the class of solutions of the higher order problem we have a one-to-one correspondence between solutions of (17.1) and solutions of (17.1)'. Thus, if u is a solution of (17.1) then $U = (u, D_t u, \ldots, D_t^{\ell-1}u)$ is a solution of (17.1)' and conversely, if $U = (u, u_1, \ldots, u_{\ell-1})$ is a solution of (17.1)' then u is a solution of (17.1) and $u_j = D_t^j u$. We shall refer to two such related solutions u and U as companion solutions.

When studying the asymptotic behavior of solutions special exponential solutions play an important role. We recall (see the Introduction) that an exponential solution of (17.1)' is a solution E(t) of the form:

$$(17.2) \qquad E(t) = e^{i\lambda_o t} P(t)$$

where P(t) is a polynomial in t with coefficients in $\mathcal{B}_p'$ (more precisely in $\mathcal{D}_A$). A necessary and sufficient condition for E(t) to be a non-zero exponential solution with P a polynomial of degree $s-1 \geq 0$,

$$(17.2)' \qquad P(t) = \Phi_s + it\Phi_{s-1} + \ldots + \frac{(it)^{s-1}}{(s-1)!}\Phi_1 , \qquad (\Phi_1 \neq 0) ,$$

is that λ_o be an eigenvalue of A and $(A-\lambda_o)\Phi_1 = 0$, $(A-\lambda_o)\Phi_k = \Phi_{k-1}$, $k = 2,\ldots,s$. We also recall that if E(t) is an exponential solution with P(t) given by (17.2)' and if we let

$$P_k(t) = \Phi_{s-k} + it\Phi_{s-k-1} + \ldots + \frac{(it)^{s-1-k}}{(s-1-k)!}\Phi_1 ,$$

then $E_k(t) = e^{i\lambda_o t} P_k(t)$ for $k = 1,\ldots,s-1$ are also exponential solutions called the associates of P.

Similarly one calls a solution $e(\cdot,t)$ of the higher order equation (17.1) an exponential solution if it has the form:

$$(17.3) \qquad e(\cdot,t) = e^{i\lambda_o t} p(\cdot,t)$$

where $p(\cdot,t)$ is a polynomial in t with coefficients belonging to $H_{2m,L_p}(G;\{B_j\})$. It is readily checked that if $e(\cdot,t)$ is an exponential solution of (17.1) then its companion $E(t) = (e,D_t e,\ldots,D_t^{\ell-1}e)$ is an exponential solution of (17.1)' with the same exponent λ_o and vice versa. The degree of the polynomial p minus 1 is called again the index of the exponential solution

(17.3). One defines the associates of e in the following manner: let E be the companion of e and let E_k, $k = 2,\dots,s$ be its associates, then the associates of e are the companions e_k of E_k.

The exponent λ_o of an exponential solution e is always an eigenvalue of A or (see previous section) an eigenvalue of the higher order eigenvalue problem:

$$(17.4)\quad \mathcal{A}(x;D_x,\lambda_o)\phi = 0\ ,\qquad \phi \in H_{2m,L_p}(G;\{B_j\})\ ,\qquad \phi \neq 0\ .$$

Using Theorem 5.4 we observe that the sequence of exponents $\{\lambda_j\}$ of all exponential solutions is a discrete set. Furthermore, if $\delta > 0$ is sufficiently small the double angle $|\arg(\pm\lambda)| \leq \delta$ contains only a finite number of eigenvalues, so that in particular there are only a finite number of λ_j in every strip $|\mathrm{Im}\ \lambda| \leq K$. We also note that the space of exponential solutions with the same exponent λ_o is finite dimensional, and that the index of an exponential solution never exceeds the order of λ_o considered as a pole of $R(\lambda;A)$.

Theorem 5.4 shows that the resolvent $R(\lambda;A)$ possesses the various properties required in the different asymptotic theorems established in Chapters 1 and 2 . Thus all these results apply to solutions U of (17.1)'. We shall confine ourselves here to Theorem 2.3 (in §5) which gives the strongest conclusions and reformulate these for solutions u of the higher order problem (17.1).

It will be convenient from now on to assume that all solutions U(t) of (17.1) (in $t > 0$) which we consider are also strongly

continuous for $t \geq 0$ (this could be achieved if necessary by translation without affecting the asymptotic properties). For solutions u of (17.1) this means that $D_t^j u$ ($j = 0,\ldots,\ell-1$) considered as functions with values in $H_{2m-jd,L_p}(G)$ are strongly continuous for $t \geq 0$. We further restrict our attention to solutions which grow at most exponentially as $t \to +\infty$. More precisely, we shall say that a solution U of (17.1)' belongs to the class $\mathcal{L}_{\omega,q}$ where ω is a real number and $1 \leq q \leq \infty$, if $e^{\omega t}\|U(t)\| \in L_q(0,\infty)$ when $1 \leq q < \infty$, while $e^{\omega t}\|U(t)\| = o(1)$ as $t \to +\infty$ when $q = \infty$. Similarly we shall say that a solution u of the higher order problem belongs to $\tilde{\mathcal{L}}_{\omega,q}$ if its companion belongs to $\mathcal{L}_{\omega,q}$, that is

$$e^{\omega t}\|D_t^j u(\cdot,t)\|_{2m-jd,L_p} \in L_q(0,\infty) \quad \text{for } j = 0,\ldots,\ell-1 \;; \quad 1 \leq q < \infty$$

$$e^{\omega t}\|D_t^j u(\cdot,t)\|_{2m-jd,L_p} = o(1)$$

$$\text{as} \quad t \to +\infty \quad \text{for } j = 0,\ldots,\ell-1 \quad \text{if} \quad q = \infty \;.$$

With each solution u belonging to some class $\tilde{\mathcal{L}}_{\omega,q}$ we associate a formal "Fourier" series:

$$u \sim \sum_{k=1}^{\infty} e_k \tag{17.5}$$

where $e_k = e^{i\lambda_k t} p_k$ are exponential solutions with different exponents λ_k satisfying $\operatorname{Im}\lambda_k > \omega$. The definition of this formal expansion is as follows: consider the companion U of u and define its formal Fourier series as in §5. That is, consider the vector valued function $R(\lambda;A)U(0)$ which (using Theorem 5.4) is a

meromorphic function of λ with poles at the eigenvalues of A (actually the function is regular for Im $\lambda \leq \omega$). Let $\{\lambda_k\}_1^\infty$ be the sequence of eigenvalues of A situated in the half-plane Im $\lambda > \omega$, and let $E_k(t) = e^{i\lambda_k t} P_k(t)$ be the exponential solution which is the residue of $R(\lambda;A)U(0)$ at $\lambda = \lambda_k$. The series $\sum_1^\infty E_k(t)$ was called by us the formal Fourier expansion of U. If, now, $E_k = (e_k, D_t e_k, \ldots, D_t^{\ell-1} e_k)$ we define $\sum_1^\infty e_k$ to be the formal Fourier series of u and write (17.5). It will be assumed in the following that the Fourier series is arranged so that Im λ_k forms a non-decreasing sequence.

The main asymptotic result here is

<u>Theorem 5.6</u>: <u>Let</u> u <u>be a solution of</u> (17.1) <u>belonging to some class</u> $\tilde{\mathcal{L}}_{\omega,q}$ $(1 \leq q \leq \infty)$. <u>Then</u>:

(i) u <u>considered as a function of</u> t <u>with values in the Banach space</u> $H_{2m,L_p}(G;\{B_j\})$ <u>is analytic for</u> $t > 0$. <u>Moreover there exists</u> $\delta > 0$ <u>depending on</u> A <u>such that</u> u <u>can be continued analytically into the angle</u> $|\arg t| < \delta$ <u>in the complex</u> t-<u>plane</u>.

(ii) <u>The formal Fourier series</u> (17.5) <u>of</u> u <u>is an asymptotic expansion in the following sense</u>: <u>Let</u> J <u>be a half-strip in the complex plane of the form</u>: $|\text{Im } t| \leq K$, $\text{Re } t \geq K \cot \delta + 1$. <u>Given a number</u> $a > \omega$ <u>let</u> $\lambda_1, \ldots, \lambda_N$ <u>be the eigenvalues (exponents) in the strip</u> $\omega < \text{Im } \lambda < a$. <u>Let</u> $\varepsilon > 0$ <u>be such that</u> Im $\lambda_k < a-\varepsilon$ <u>for</u> $k = 1, \ldots, N$. <u>Then the following inequality holds for</u> $t \in J$:

$$\|D_t^j(u(\cdot,t) - \sum_{k=1}^N e_k(\cdot,t))\|_{2m,L_p(G)}$$

(17.6)
$$\leq \text{constant}\left(\sum_{i=0}^{\ell-1} \|D_t^i u(\cdot,0)\|_{2m-(i+1)d,L_p}\right) e^{-(a-\varepsilon)\text{Re } t}$$

for $j = 0,\ldots$. Here the constant depends on a, ε, j, δ, K (and in general on A) but not on the solution u.

Proof: It will suffice to establish (17.6) for $j = 0$, the estimates for the other derivatives follow easily from this by Cauchy's formula using the analyticity. One can further assume without loss of generality that $q = 1$. Indeed, if $u \in \tilde{\mathcal{L}}_{\omega,q}$ then $u \in \tilde{\mathcal{L}}_{\omega',1}$ for every $\omega' < \omega$. Thus assuming we have established the theorem for this class it would follow that (17.6) holds with the difference that the Fourier expansion of u might now contain additional exponentials with exponents in the strip $\omega' < \operatorname{Im} \lambda \leq \omega$. This, however, is ruled out by the assumption $u \in \tilde{\mathcal{L}}_{\omega,q}$. Finally, we can also assume that $\omega = 0$ since we can replace u by $v = e^{\omega t}u$.

Summing up it suffices to establish the theorem for $j = 0$ and $u \in \tilde{\mathcal{L}}_{0,1}$. Let $U = (u, D_t u, \ldots, D_t^{\ell-1} u)$, then U is a solution of (17.1)' which belongs to L_1 on $t \geq 0$. We now apply to U Theorem 2.3 of the abstract theory, with $X = \mathcal{B}_p'$, $Y = \mathcal{B}_p$. By Theorem 5.4, $R(\lambda;A)$ (λ non-eigenvalue) takes Y boundedly onto X and is analytic in λ. Also, if $\delta > 0$ is the constant appearing in Theorem 5.4 then there are only a finite number of eigenvalues in the double angle $|\arg(\pm\lambda)| \leq \delta$ and for λ sufficiently large, $|\lambda| \geq c$, in the double angle: $|R(\lambda;A)|_X = O(|\lambda|^{\ell-1})$ $(\lambda \to \infty)$. Thus, all the conditions of Theorem 2.3 hold with $\theta_1 = \theta_2 = \delta$ and α any small positive number. From the theorem it follows then that $U(t)$ is analytic in the angle $|\arg(t-\alpha)| < \delta$ (and, since $\alpha > 0$ is arbitrary, in $|\arg t| < \delta$) where it satisfies

$$(17.7)\quad |U(t) - \sum_1^N E_k(t)|_X \le \text{constant } |U(0)|_Y \frac{e^{-(a-\varepsilon)\mu}}{\mu} e^{c|\mathrm{Im}\, t|}$$

where $E_k = (e_k, D_t e_k, \ldots, D_t^{\ell-1} e_k))$ and $\mu = \mathrm{Re}\, t - \alpha - (\mathrm{Im}\, t)\cot\delta$ and the constant depends only on δ, a, ε and A. In particular, restricting t to the strip J we obtain:

$$(17.7)'\quad |U(t) - \sum_1^N E_k(t)|_X \le \text{constant } |U(0)|_Y e^{-(a-\varepsilon)\mathrm{Re}\, t}$$

with constant dependence as in the theorem. The analyticity of u and (17.6) (with $j = 0$) follows now immediately by taking first components of U etc., using the analyticity of U and (17.7)'. This completes the proof.

Remark: For standard elliptic problems (and probably also in the general case) the growth restrictions on u in Theorem 5.6 could be relaxed considerably. Indeed in this case ($\ell = 2m$) it suffices to assume that $e^{\omega t}\|u(\cdot,t)\|_{L_p(G)} \in L_q(0,\infty)$ for some $1 \le q < \infty$ or $e^{\omega t}\|u(\cdot,t)\|_{L_p(G)} = o(1)$ in order that the conclusion of the theorem should hold. To see this we first note that by the same reduction as used in the proof of Theorem 5.6 it suffices to consider the special case $\|u(\cdot,t)\| \in L_1(0,\infty)$ $(\omega = 0,\ q = 1)$. Now, denoting by Γ_a^b the section of the cylinder Γ contained in $a < t < b$, it follows (essentially) from the a priori L_p estimates established in Agmon, Douglis, Nirenberg [1] that the following majorizations hold (the norms are in all (x,t) variables):

$$(17.8)\quad \|u(x,t)\|_{2m,L_p(\Gamma_k^{k+1})} \le \text{constant } \|u\|_{L_p(\Gamma_{k-1}^{k+2})},$$

for $k = 1,2,\ldots,$ the constant being independent of k. Summing (17.8) with respect to all k we find that our assumption implies that $u \in \tilde{\mathcal{L}}_{0,1}$, and hence Theorem 5.6 holds under the seemingly much weaker growth restriction.

As an immediate corollary of Theorem 5.6 we obtain a

Phragmén-Lindelöf Principle: Let u be a solution of (17.1) belonging to some class $\tilde{\mathcal{L}}_{\omega,q}$. Let $\Delta > 0$ be the distance of the subset of eigenvalues λ_k of (17.4) situated in the half-plane $\operatorname{Im} \lambda > \omega$ from the boundary $\operatorname{Im} \lambda = \omega$. Then, for every $\varepsilon > 0$ and $j = 0,1,\ldots,$:

$$\|D_t^j u(\cdot,t)\|_{2m,L_p(G)} = O(e^{-(\omega+\Delta-\varepsilon)t}) \quad \text{as} \quad t \longrightarrow +\infty .$$

We conclude this section with an application of the abstract Weinstein primciple of §6 concerning solutions defined on the whole real axis.

Theorem 5.7: Let $u(x,t)$ be a solution of the corresponding boundary value problem (17.1) defined, however, in the whole cylinder Γ. Suppose that u is of class $\tilde{\mathcal{L}}_{\omega,q}$ for $t \geq 0$ and also that $u(x,-t)$ is of class $\tilde{\mathcal{L}}_{\omega_1,q_1}$ for $t \geq 0$. The following holds:

(i) If $\omega + \omega_1 \leq 0$, then $u = 0$.

(ii) If $\omega + \omega_1 > 0$ then u is a finite sum of exponential solutions $e_k = e^{i\lambda_k t} p_k$ where λ_k is an eigenvalue of (17.4) satisfying $-\omega_1 < \operatorname{Im} \lambda_k < \omega$.

The theorem is an immediate consequence of Theorem 2.5 and Theorem 5.4 applied to the companion solution $U = (u,D_t u,\ldots,D_t^{\ell-1}u)$

noting (as in the proof of Theorem 5.6) that without loss of generality we may assume $q = q_1 = 1$.

The remark which follows the proof of Theorem 5.6 applies also to Theorem 5.7. Namely, for ordinary elliptic problems, and probably also in the general case, one can replace the growth restrictions in the theorem by the following weaker assumptions: $e^{\omega t}\|u(\cdot,t)\|_{L_p(G)} \in L_q(0,\infty)$ for some $1 \leq q < \infty$ or $e^{\omega t}\|u(\cdot,t)\| = o(1)$ for $t \longrightarrow +\infty$, and $e^{\omega_1 t}\|u(\cdot,-t)\|_{L_p(G)} \in L_{q_1}(0,\infty)$ for some $1 \leq q_1 < \infty$ or $e^{\omega_1 t}\|u(\cdot,-t)\|_{L_p} = o(1)$ for some $t \longrightarrow +\infty$.

We observe that if (ii) of Theorem 5.7 holds and if there are no eigenvalues in the strip $-\omega_1 < \operatorname{Im} \lambda < \omega$ it follows that $u \equiv 0$. This (and (i)) gives a kind of a Phragmén-Lindelöf principle for solutions u defined in the whole infinite cylinder. On the other hand, if there are eigenvalues in the strip their number is necessarily finite. Hence, the space of solutions satisfying the growth restrictions of the theorem for fixed ω, ω_1 is finite dimensional, the dimension being equal to the sum of dimensions of the generalized eigenspaces of A corresponding to the eigenvalues in the strip.

Remark: One might expect that the space of solutions of an elliptic problem in a doubly infinite cylinder with reasonable boundary conditions and behaving reasonably at infinity is finite dimensional even for operators whose coefficients are allowed to depend on t — provided, say, the operator is uniformly elliptic. This is, however, not the case, as may be seen with the aid of a result of A. Plis. Plis [1] has constructed an elliptic operator

T with real leading coefficients for which there is a nontrivial solution v of $Tv = 0$ with compact support. We may suppose that v vanishes outside the unit sphere. Let now Γ be the doubly infinite cylinder over a unit sphere, and let L be an elliptic operator in Γ with coefficients periodic (period 2π) in the x_{n+1} direction, parallel to the sides of the cylinder, and such that $L = T$ in the unit sphere about the origin. Such a uniformly operator L is easily constructed. For j = an integer let $v_j = v(x_1,\ldots,x_{n+1} - 2\pi j)$, where v is the solution constructed by Plis. The functions v_j all have zero Cauchy data on the sides of the cylinder, have compact support, and are linearly independent.

18. Completeness of exponential solutions

Continuing the investigations of the previous section we turn now to the problem of completeness of exponential solutions. We shall make use of the abstract completeness results (Theorems 2.7, 2.8) and the information on the lower order of $R(\lambda;A)$ formulated as Theorem 5.4'. Since this theorem was established only for $p = 2$ we shall impose this restriction in the following discussion.

We shall denote by W_θ, θ a real number, the triplet:

$$(18.1) \qquad (\mathcal{A}(x;D_x,e^{i\theta}D_t),\{B_j(x;D_x)\};\Gamma)$$

where $W_o = (\mathcal{A}(x;D_x,D_t),\{B_j(x;D_x)\};\Gamma)$ is the given regular weighted elliptic boundary value problem of order type $(2m,\ell)$. Our completeness results will be proved under

Hypothesis C: Either $d \geq n$ or, if $d < n$, assume that there exist numbers $0 = \theta_o < \theta_1 < \dots < \theta_s < \theta_{s+1} = \pi$ such that

(18.2) $$\theta_{j+1} - \theta_j \leq \frac{d}{n}\pi \qquad \text{for} \quad j = 0,\dots,s ,$$

and such that W_{θ_j} is a regular weighted elliptic problem for $j = 1,\dots,s$.

We observe that hypothesis (18.2) is of an algebraic nature. For W_θ $(0 < |\theta| < \pi)$ to be a regular problem one needs simply that $\mathcal{A}(x;D_x,e^{i\theta}D_t)$ be also elliptic (with the additional "roots assumption" for $n = 1$) and that this operator together with the boundary system $\{B_j(x;D_x)\}$ satisfy the algebraic complementing condition on $\partial\Gamma$. We also add the following

Remark: If W_θ is a regular problem then also $W_{\theta+\pi}$ is a regular problem (the algebraic conditions remain invariant under the substitution $t \to -t$). Also, if $W_{\theta*}$ is regular then it follows by continuity that W_θ is regular for all θ with $|\theta-\theta^*| < \varepsilon$, $\varepsilon > 0$ sufficiently small. From the last observation it follows that if Hypothesis C holds then without loss of generality we may assume that (18.2) hold as strict inequalities. Also in the trivial situation of the hypotheses $d \geq n$, we can always choose a $\theta_1 > 0$ sufficiently small so that W_{θ_1} be regular so that also in this case (18.2) holds for the triplet $0 = \theta_o < \theta_1 < \theta_2 = \pi$. In the Following when the hypotheses will be used it will be assumed that the θ_j were modified so as to have strict inequalities and that we also have (18.2) in the trivial situation as explained.

We pass now to the completeness theorems.

<u>Theorem 5.8</u>: <u>Let</u> $(\mathcal{A}, \{B_j\}; \Gamma)$ <u>be a regular weighted elliptic boundary value problem which satisfies Hypothesis C. Let</u> u <u>be a solution of the homogeneous boundary value problem</u> (17.1) <u>in the semi-infinite cylinder</u> Γ^+ <u>in the sense described previously</u>. <u>Assume that</u> $u = u(\cdot,t)$ <u>belongs to</u> $\tilde{\mathcal{L}}_{0,1}$ <u>on</u> $t \geq 0$ [1] <u>and let</u>

$$u(\cdot,t) \sim \sum_{j=1}^{\infty} e_j(\cdot,t)$$

be the formal Fourier expansion of u, e_j being an exponential solution of index m_j. Then, given $\varepsilon > 0$, $N > 0$ and an integer $K \geq 0$, there exists a linear combination ψ of exponential solutions of the form:

$$\psi(\cdot,t) = \sum_{j=1}^{M} \sum_{k=0}^{m_j-1} a_{jk} D_t^k e_j \tag{18.3}$$

such that

$$\|D_t^j(u(\cdot,t) - \psi(\cdot,t))\|_{2m,L_p(G)} \leq \varepsilon e^{-Nt} \tag{18.3$'$}$$

for $t \geq \varepsilon$ and $j = 0,1,\ldots,K$.

<u>Proof</u>: Let $U = (u, D_t u, \ldots, D_t^{\ell-1} u)$. Then, $U(t)$ is a solution of (17.1)' or

$$D_t U - AU = 0 \quad \text{for} \quad t > 0 .$$

Also $U(t) \in L_1(0,\infty)$ by our assumption on u. Using Theorem 5.4' we find further that the lower order $\underline{\omega}(A)$ of $R(\lambda;A)$ does not exceed $\frac{n}{d}$. Using the Hypothesis and the remark made above, there

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[1] This can be replaced by the weaker assumption $u(x,t) \in L_1(\Gamma^+)$.

exist numbers $0 = \theta_o < \theta_1 < \dots < \theta_k < \theta_{k+1} = \pi$ such that $\theta_{j+1} - \theta_j < \frac{d}{n}\pi$ and such that W_{θ_j} is a regular weighted elliptic boundary value problem for $j = 0,1,\dots,k+1$. Making use of Theorem 5.4 (applied to W_{θ_j}) it follows that along each of the rays: $\arg\lambda = \theta_j$ $(j = 0,\dots,k+1)$ the resolvent $R(\lambda;A)$ exists for λ sufficiently large and satisfies:

$$(18.4) \qquad |R(\lambda;A)|_X = O(|\lambda|^{\ell-1}) \qquad (\arg\lambda = \theta_j\ , \quad \lambda \longrightarrow \infty)\ .$$

Thus U and A satisfy the conditions of Theorem 2.7, and it then follows that if $\sum E_j$ is the formal Fourier expansion in exponential solutions of U, then there exists a finite sum of exponential solutions $\underline{\Psi}$ of the form:

$$(18.5) \qquad \underline{\Psi}(t) = \sum_{j=1}^{M} \sum_{k=0}^{m_j-1} a_{jk} D_t^k E_j\ ,$$

m_j being the index of E_j, such that

$$(18.5)' \qquad |D_t^j(U(t) - \underline{\Psi}(t))|_{\mathcal{B}_p'} \le \varepsilon e^{-Nt} \quad \text{for} \quad t \ge \varepsilon\ , \quad j = 1,\dots,K\ .$$

(Here one can take α and the τ_j, $j = 1,\dots,K$, of Theorem 2.7 arbitrarily small.) The desired result (18.3)' is an immediate consequence of (18.5)' and (18.5) by restriction to first components of the vectors U, $\underline{\Psi}$.

A very similar completeness result holds for solutions in finite cylinders. The only difference is that now one must take all exponential solutions.

Theorem 5.8': Let $(\mathcal{A},\{B_j\};\Gamma)$ satisfy the conditions of Theorem 5.8. Let u be a solution of the corresponding problem (17.1) for a finite cylinder Γ_{-T}^{T} (the part of Γ in $|t| < T$). Then given $\varepsilon > 0$ and an integer $K \geq 0$ there exists a finite sum of exponential solutions ψ such that

$$\|D_t^j(u(\cdot,t)-\psi(\cdot,t))\|_{2m,L_p(G)} \leq \varepsilon \quad \text{for } |t| \leq T-\varepsilon\ , \quad j = 0,\ldots,K\ .$$

The proof of the theorem is exactly the same as the proof of Theorem 5.8 using instead Theorem 2.8 of the abstract theory and using the previous observation that regularity of W_θ implies regularity of $W_{\theta+\pi}$, so that $R(\lambda;A)$ actually satisfies (18.5) along the straight lines $\lambda = \rho^{i\theta_j}$, $-\infty < \rho < \infty$, $|\rho|$ sufficiently large and $\longrightarrow \infty$. We note in passing also the following result which uses the same observation.

Theorem 5.8": Let $(\mathcal{A},\{B_j\};\Gamma)$ satisfy the conditions of Theorem 5.8 and let A be the corresponding operator acting on $\mathcal{B}_2'$. Then the generalized eigenelements of A are complete in $\mathcal{B}_2$.

This result follows easily from the properties of the resolvent. For a proof see Agmon [2, Theorem 3.2].

Let us consider some special concrete case. Consider first the case of two dimensional weighted elliptic boundary value problems, i.e. $n = 1$. Then $\frac{n}{d} \leq 1$. Hence, without any additional assumptions we have completeness of exponential solutions in this case in the sense of Theorems 5.8, 5.8'.

Another special case is when $\mathcal{A}$ is elliptic of order 2m and such that, at any point, its principal part may be expressed as a product of m second order operators:

$$\prod_{k=1}^{m} (D_t^2 + \sum a_{ij}^{(k)} D_{x_i} D_{x_j})$$

with real coefficients $a_{ij}^{(k)}$. Then the operator

$$\prod_{k=1}^{m} (e^{2i\theta} D_t^2 + \sum a_{ij}^{(k)} D_{x_i} D_{x_j})$$

are elliptic for every $\theta \neq \pm \frac{\pi}{2}$. If the boundary system $\{B_j\}$ is such that the complementing condition holds with respect to these operators (this will be the case, for instance, for the Dirichlet boundary conditions, or if $B_j' = (\frac{\partial}{\partial\rho})^{k_o+j}$ where ρ is a variable non-tangential direction on ∂G, and $0 \leq k_o \leq m-1$) then all the problems W_θ for $\theta \neq \pm \frac{\pi}{2}$ are regular problems, so that Hypothesis C holds. Consequently we have completeness of exponential solutions as formulated in Theorems 5.8 and 5.8'.

We mention that in the special case of the bi-harmonic equation in two variables (Dirichlet boundary values) a completeness result for a semi-infinite strip was previously established by Smith [1] (also Lax [2] for a different proof). This result is of course contained in each of the two special situations described above.

19. Stability and exponential decay of solutions at infinity

Let $(\mathcal{A}, \{B_j\}_1^m; \Gamma)$ be a regular weighted elliptic boundary value problem order type $(2m, \ell)$ with, as before, $d = \frac{2m}{\ell}$ an integer, and $\mathcal{A}$ given by (16.1).

In this section we are interested in the behavior as $t \to \infty$ of solutions of an elliptic problem in the half cylinder Γ^+

$$(19.1) \quad \tilde{\mathcal{A}}(x,t;D_x,D_t)u = f\ , \qquad B_j u = 0 \ \text{ on side of cylinder}$$

where $f \to 0$ in some sense as $t \to \infty$ and where the coefficients of $\tilde{\mathcal{A}}$ may depend on t, but so that $\tilde{\mathcal{A}}$ approaches $\mathcal{A}$ as $t \to \infty$.

In general, in treating such questions of limiting behavior one should also permit the operators B_j to depend on t and permit $f(x,t)$ to tend to some function $g(x)$ not necessarily zero. However we shall confine ourseles to the more special situation; by suitable modifications of the methods employed here it is also possible to treat some more general cases. A. Friedman has made an extensive study of behavior at infinity of solutions of equations of the form

$$\frac{\partial u}{\partial t} - \mathcal{A}(x;t;D_x)u = f$$

where $\mathcal{A}$ is strongly elliptic, as well as certain nonlinear equations, see Friedman [1-3].

In practice (say for well posed problems) it is often possible to obtain exponential bounds for solutions of (19.1) and their derivatives, and we shall assume that such bounds maintain; in fact, after multiplication of u by a suitable factor $e^{-\sigma t}$ we may suppose that the solutions are bounded (in some sense) at infinity.

We shall apply the results of §3, in particular Theorem 1.2" which requires however that the spaces X and Y be Hilbert spaces. We shall therefore consider the spaces $X = \mathcal{B}'_p$, $Y = \mathcal{B}_p$ with $p = 2$ and denote the L_2 norm in the x variables of a function v by $\|v\|_{L_2}$ and set

$$\sum_{\substack{j \leq \ell \\ i \leq 2m-jd}} \|D_x^i D_t^j u\|_{L_2} = |||u||| .$$

We express equation (19.1) as well as the condition that the coefficients of $\tilde{\mathcal{A}}$ tend to those of $\mathcal{A}$ by the inequality: for every $t > 0$, $u \in H_{2m,L_2}(G;\{B_j\})$

$$(19.2) \qquad \|\mathcal{A}(x;D_x,D_t)u\|_{L_2} \leq \frac{c}{(1+t)^k} |||u||| + b(t)$$

where $b(t)$ is scalar valued. Here c is constant and k is a non-negative integer; in some cases we shall take $k = 0$. If we introduce the analogous system as in §16 setting $u = u_o$, $U = (u_o, D_t u_o, \ldots, D_t^{\ell-1} u_o)$ and defining A by (16.3)', the inequality takes the form, with $L = D_t - A$

$$|LU|_Y \leq \frac{c}{(1+t)^k} \left(|U|_X + \|D_t^\ell u_o\|_{L_2}\right) + b(t)$$

with a different constant c. Since $\|D_t^\ell u\|_{L_2} \leq |LU|_Y + \text{constant } |U|_X$, for small c the inequality may be written in the form

$$(19.3) \qquad |LU|_Y \leq \frac{c}{(1+t)^k} |U|_X + b(t)$$

with c and $b(t)$ slightly changed. We note also that then the inequality

$$(19.4) \qquad |||u||| \leq C(|U|_X + b(t))$$

also holds, with some positive constant C independent of u.

We first present a result in which a solution which is bounded by an exponential is shown to decay faster than that exponential. In the following $p > 1$ is a fixed finite number. We shall use $\tilde{R}(\lambda)$ as defined at the end of §16.

Theorem 5.9: Assume that the boundary value problem

$$\mathcal{A}(x;D_x,\lambda)u = 0\ , \qquad u \in H_{2m,L_2}(G;\{B_j\})$$

has no nontrivial solutions for λ in the strip $\varepsilon < \operatorname{Im}\lambda \leq \varepsilon_1$, i.e. there are no eigenvalues in the strip. By Theorem 5.4 it follows that there are no eigenvalues in a larger strip $\varepsilon < \operatorname{Im}\lambda < a$, for $a > \varepsilon_1$. Assume that on the line $\operatorname{Im}\lambda = \varepsilon$, $R(\lambda;A)$ has poles (necessarily finite in number) of maximal order $k \geq 0$. Assume that u is a solution of (19.2) with $e^{a't}b(t) \in L_p$ on $t > 0$ for some a', $\varepsilon < a' < a$. Assume furthermore that $e^{\varepsilon t}|||u|||$ belongs to L_p. Then, if c is sufficiently small, the inequality

$$(19.5)\qquad \int_0^\infty \left|e^{a't}|||u|||\right|^p dt \leq C\int_0^1 |||u|||^p dt + C\int_0^\infty |e^{a't}b(t)|^p dt$$

holds, for some constant C.

Proof: By our hypothesis $e^{\varepsilon t}|U|_X$ belongs to L_p. If we set $e^{\varepsilon t}U = V$, then by (19.3), V satisfies

$$|(L+i\varepsilon)V|_Y \leq c|V|_X + e^{\varepsilon t}b(t)\ .$$

With the aid of Theorem 5.4 we see that the operator $L + i\varepsilon$ satisfies all the conditions of Theorem 1.2", with the $Q_j = 0$ and

all the $P_j = 0$ except $P_1 = I$. Applying the theorem we see that $e^{(a'-\varepsilon)t}|V|_X \in L_p$ and that

$$\int_0^\infty \left|e^{a'-\varepsilon)t}|V|_X\right|^p dt \le C_1 \int_0^1 |V|_X^p dt + C_1 \int_0^\infty \left|e^{a't}b(t)\right|^p dt .$$

This, together with (19.4) yields the desired inequality.

Remarks: 1) In applying this theorem to (19.1) observe that if $k = 0$ the coefficients of $\mathcal{A}(x,t;D_x.D_t)$ are not required to converge to the corresponding coefficients of $\mathcal{A}(x;D_x,D_t)$, but merely to differ little from them for large t.

2) We use L_2 norms in the x directions but L_p norms in the t direction. This may be useful in case, say, $b(t) = O(t^{-\alpha})$ for some $\alpha > 0$. Then b(t) may not belong to L_2 but will belong to L_p for p sufficiently large.

3) From (19.5) it follows easily that as $t \to \infty$

$$\|D_x^i D_j^t u\| = o(e^{-a't}) \qquad \text{for} \quad j < \ell \ ; \quad i \le 2m-(j+1)d .$$

In particular if $b(t) = 0$ we conclude that these norms are $o(e^{-a't})$ for any $a' < a$.

4) Suppose that u is a solution of (19.1) in the full cylinder $-\infty < t < \infty$, and suppose that $\tilde{\mathcal{A}}$ is elliptic, so that $d = 1$. In general one wants to know if the space of solutions, satisfying suitable growth conditions at $t = \pm\infty$, is finite dimensional. We have shown in the remark at the end of §17 that this need not be the case, even if $\tilde{\mathcal{A}}$ is uniformly elliptic. However, if the coefficients of $\tilde{\mathcal{A}}$ approach with sufficient

rapidity the corresponding coefficients of $\mathcal{A}_{\pm}(x;D_x,D_t)$ as $t \to \pm\infty$ (where $\mathcal{A}_{\pm}$ is an elliptic operator with coefficients independent of t such that $(\mathcal{A}_{\pm},\{B_j\};\Gamma)$ is a regular elliptic boundary value problem) then the space of solutions u of (19.1) with suitable growth conditions at $\pm\infty$ can be shown to be finite dimensional. We illustrate this with the following result. We shall assume that $\tilde{\mathcal{A}}$ is uniformly elliptic with bounded coefficients and continuous leading coefficients, and we use the notation of Theorem 5.9.

With a and b fixed real numbers, consider the space of solutions of (19.1) on $-\infty < t < \infty$ for which

$$\tilde{|}u|^2 = \int_0^\infty e^{2at}\|u\|^2dt + \int_{-\infty}^0 e^{-2bt}\|u\|^2dt$$

is finite. Using the results of Agmon, Douglis, Nirenberg [1] (see (17.8)) one verifies that also

$$(19.6) \qquad \int_0^\infty e^{2at}|||u|||^2dt + \int_{-\infty}^0 e^{-2bt}|||u|||^2dt < \text{constant } \tilde{|}u|^2$$

with the constant independent of u. In order to prove the finite dimensionality of the space it suffices to show that out of a sequence of solutions with bounded $\tilde{|}\ |$ norm it is possible to select a subsequence converging in the norm. From (19.6) it is easily seen that there is a subsequence converging in the norm

$$\left[\int_0^T e^{2at}\|u\|^2dt + \int_{-T}^0 e^{-2bt}\|u\|^2dt\right]^{1/2}$$

for any finite T. The proof will then be complete if we can establish a faster decay of solutions at $\pm\infty$ than that implied by the fact that $\tilde{|u|}$ is finite — if, for instance, we can show that

$$(19.7) \qquad \int_0^\infty e^{2a't}\|u\|^2 dt + \int_{-\infty}^0 e^{-2b't}\|u\|^2 dt \leq \text{constant } \tilde{|u|}^2$$

for certain constants a', b' with $a' > a$, $b' > b$.

We now add additional assumptions enabling us to derive such an inequality.

<u>Assumptions</u>: On the line Im $\lambda = a$, $\tilde{R}(\lambda, \mathcal{A}_+)$ has poles (necessarily finite in number) of maximal order k_+, and on Im $\lambda = b$, $\tilde{R}(\lambda, \mathcal{A}_-)$ has poles of maximal order k_-. Assume furthermore that as $t \longrightarrow \pm\infty$ the difference of each coefficient of $\tilde{\mathcal{A}}$ from the corresponding coefficient of $\mathcal{A}_\pm$ is bounded in absolute value by

$$\frac{c}{(1\pm t)^{k_\pm}} .$$

Under these assumptions we find from Theorem 5.9 that if c is sufficiently small then (19.7) follows, indeed one has

$$\int_0^\infty e^{2a't}|||u|||^2 dt + \int_{-\infty}^0 e^{-2b't}|||u|||^2 dt$$

$$\leq \text{constant (left hand side of (19.6))}$$

$$\leq \text{constant } \tilde{|u|}^2 .$$

Hence we conclude that, under these assumptions, the space of solutions is dinite dimensional.

More generally, consider solutions of a homogeneous elliptic boundary value problem in an unbounded domain consisting of a bounded region plus a finite number N of tubes going to infinity each of which may be mapped onto a semi-infinite cylinder Γ_j, $j = 1,\ldots,N$ by "smooth" mappings and suppose that the solutions satisfy some reasonable growth properties at infinity. Suppose furthermore (this being very restrictive in practice) that in each Γ_j the problem takes the form of (19.1) with the B_j <u>independent of</u> t, and that the coefficients of $\tilde{\mathcal{A}}$ in Γ_j approach those of some operator $\mathcal{A}_j$ with sufficient rapidity, where $(\mathcal{A}_j, \{B_j\}_1^m; \Gamma_j)$ is a regular elliptic problem. Then with the aid of Theorem 5.9 it may be possible to conclude that the space of solutions is finite dimensional.

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